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A Historical and Critical Review

of Mathematical Science

EDITED BY

EARLE R. HEDRICK

DAVID R. CURTISS WILLIAM R. LONGLEY

WITH THE ASSISTANCE OF

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JOHN R. KLINE HENRY L. RIETZ

CAROLINE E. SEELY

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AN ANNOUNCEMENT

With the present issue, the Bulletin will assume a somewhat different appearance. The changes involved have been under discussion for over a year, and were finally formulated and authorized by the Council of the Society at its meeting at Bethlehem, Pennsylvania, on December 27, 1929.

The only change in topical arrangement will be that the abstracts of papers will be printed under a separate heading rather than in the reports of meetings. The abstracts will be numbered serially throughout the year; references to them in the reports of meetings will include not only this serial number, but also the number of the volume and the number of the issue in which the abstract is to be found, to facilitate cross-reference.

For the immediate present, this change is only a superficial one, since the abstracts that occur in this issue are those of papers read at meetings reported in the same issue. In the future, however, it is hoped that more and more of the abstracts can be printed *in advance* of the meetings at which the papers are presented, with obvious advantages to both authors and auditors. Members will be notified from time to time by the Secretary and the Associate Secretaries of the conditions under which such advance publication of their abstracts can be secured.

It is to make possible the plan just mentioned, and to have greater unity in contents, that the type of matter presented in this issue is segregated from the papers that have always formed a part of the Bulletin. Such papers, including the longer addresses, will appear as usual, but in separate issues of the Bulletin, to be published in the alternate months. The handling of printers' proofs for such papers requires a longer time and is subject to delays which cannot be controlled. The segrega-

tion of types of material which occur in the present issue will make possible much quicker publication of it.

There will be, then, twelve issues each year. The odd-numbered issues will contain the material just mentioned, as illustrated by the present issue. The even-numbered issues will contain only the papers and the list of new publications. While the alternate issues will have different appearances, it will be evident that the bound volume of the Bulletin will not differ materially from previous volumes, since the types of material will alternate much as in the past. The size of the volume will not be affected.

It is hoped that the new form will serve a useful purpose in providing quick publication of certain types of matter, notably the abstracts of papers, as mentioned above. It is hoped also that the segregation of material will be pleasing to the readers, in that the type of matter in any one issue will have greater unity and homogeneity.

Eventually, the new forms, if they meet with general approval, will constitute a framework upon which future needs and future growths may be accommodated without violent change. One demand which is reasonable and which has arisen frequently and somewhat insistently, is for considerable increase in the maximum length of abstracts. Just now, such an action is precluded both by lack of financial resources for further printing and by a natural limitation upon the desirable size of a single volume of the Bulletin. It is obvious that the two types of matter represented by the alternate issues may be made the bases of separate publications when and if the necessity arises and the means are avialable. On the other hand, the present forms do not constitute, in themselves, a serious change from previous volumes, as has been pointed out. If experience shows that they are not generally approved, the Society may return to the previous forms at any time without marked discontinuity in the series.

Comments and suggestions by members and by other readers are proper at all times. Just now, however, the editors particularly invite them, as a guide to their own action and to their future recommendations to the Council of the Society.

THE EDITORS

THE SOCIETY'S PRIZES

At the Annual Meeting of the Society held at Bethlehem, the Council adopted a report received from its Committee on the Reorganization of the Society's prizes, composed of Professors R. D. Carmichael (Chairman), Abraham Cohen, and H. H. Mitchell. The conditions of award, as established by action of the Council and Board of Trustees in accordance with these recommendations, will govern future procedure.

One reason for the reorganization was the insurmountable difficulty found by earlier committees of award of the Bôcher Memorial Prize in reaching a confident conclusion when an award is made without restriction of subject matter for memoirs considered. Another is that through the generosity of Mr. Charles A. Cole the funds in honor of his father, Professor Frank Nelson Cole, have been doubled.

It was decided to establish three prizes of two hundred dollars ($200) each, to limit the field for each prize, and to award them at five-year intervals in accordance with the following schedule:

(1) The Bôcher Prize in Analysis to be awarded at the end of 1933 for papers in analysis published during the years 1928 to 1932 inclusive;

(2) The Cole Prize in Algebra to be awarded at the end of 1934 for papers published during the years 1929–33;

(3) The Cole Prize in the Theory of Numbers to be awarded at the end of 1931 for papers published during the years 1926–30 inclusive.

If the committtee on award finds no memoir of outstanding merit at the time appointed for a given award, it will report this fact to the Council; the Council concurring, no award shall be made at that time.

The following conditions for eligibility have been adopted:

1. A memoir to be eligible for consideration must have been published in the appropriate interval of time in a journal on the editorial board of which the American Mathematical Society

was officially represented at the time of publication of the memoir.

2. The author at the time of publication of the memoir must have been a member of the American Mathematical Society.

3. The author at the time of publication of the memoir must have been not more than fifty years of age.

4. No dissertation shall be eligible for any one of the prizes.

5. No author who has once received a particular prize shall again be awarded the same prize.

6. No paper shall be eligible for more than one prize.

The Bôcher Prize in Analysis is a continuation of the Bôcher Memorial Prize (see this Bulletin, vol. 28, p. 42). The conditions of award have, however, been changed so that the subject matter is now restricted to analysis; on the other hand, the place of publication is no longer restricted to the Transactions of this Society, but may be any journal on whose editorial board the Society is represented. The two Cole prizes are in many respects a continuation of the Cole Prize in Algebra (see this Bulletin, vol. 29, p. 14), but are no longer restricted to papers offered in competition on announced topics.

The report of the Committee stressed the desirability of having prizes also in geometry and in applied mathematics. Such prizes cannot be established until special funds for them have been secured. It is hoped that the Society may soon be in a position to announce these additional awards so that one prize may be given each year.

R. G. RICHARDSON,
Secretary

THE OCTOBER MEETING IN NEW YORK

The two hundred seventy-second regular meeting of the American Mathematical Society was held at Columbia University, on Saturday, October 26, 1929, extending through the usual morning and afternoon sessions. The attendance included the following ninety-two members:

C. R. Adams, R. B. Adams, A. A. Albert, R. C. Archibald, R. G. Archibald, H. E. Arnold, Bellamy, Benton, Bernstorf, William Johnston Berry, A. B. Brown, B. H. Camp, Carrié, L. W. Cohen, Collier, L. P. Copeland, C. C. Craig, Dantzig, Dehn, Demos, Feld, Fiske, Fite, D. A. Flanders, Fort, R. M. Foster, Gill, Glenn, Gourin, Gronwall, C. C. Grove, Helliwell, Hille, Hofmann, Dunham Jackson, R. L. Jackson, Joffe, M. I. Johnson, R. A. Johnson, Kasner, Kenny, Kholodovsky, Rose Klein, Knobelauch, Kramer, Landers, Lefschetz, D. H. Lehmer, Littauer, Lotka, MacColl, Meder, F. H. Miller, A. K. Mitchell, Molina, Richard Morris, Mullins, F. H. Murray, C. A. Nelson, Newman, Oakley, K. E. O'Brien, Ore, Parsons, Pell-Wheeler, Pfeiffer, Pierpont, Pogo, Post, T. M. Putnam, Raudenbush, Reddick, C. J. Rees, R. G. D. Richardson, Ritt, Schelkunoff, Seely, Serghiesco, Shewhart, Siceloff, Simons, Smail, J. M. Thomas, W. R. Thompson, Trjitzinsky, H. E. Webb, Weida, Weisner, Whittemore, Wiener, W. A. Wilson, Margaret M. Young.

There was no meeting of the Council or of the Board of Trustees. At a business session of the Society held before the beginning of the regular afternoon session, the following amendments to the By-Laws were adopted, as recommended by the Council: Article I, Section 1, to read "The officers of the Society shall be a President, a Secretary three Associate Secretaries," and continue as before. Article, I, Section 2, to read "It shall be the duty of the President to deliver an address before the Society at the Annual Meeting one year after his term of office expires." Article VII, Section 3, second sentence to read "Special meetings of the Board of Trustees may be called by the Chairman of the Board upon three days notice of such meeting mailed to the last known post office address of each Trustee."

At the request of the Program Committee, Professor Tomlinson Fort, of Lehigh University, delivered, at the beginning of the afternoon session, an address entitled *General theory of factorial series*. This address will appear in full in an early issue of this Bulletin.

Professor Tomlinson Fort presided at the morning session relieved by Professor Einar Hille, and Professor James Pierpont at the afternoon session, relieved by Professor Dunham Jackson.

The titles and cross-references to the abstracts* of the papers read at this meeting, follow below. The papers whose abstract numbers are accompanied by the letter *t* were read by title. Mr. Borofsky was introduced by Professor Ritt, and Mr. Langdon by Professor Pierpont.

1. *The structure of pure Riemann matrices with non-commutative multiplication algebras*, by Dr. A. A. Albert. (Abstract No. 36-1-1-*t*.)

2. *A construction of rational cyclic division algebras of order sixteen*, by Dr. A. A. Albert. (Abstract No. 36-1-2.)

3. *On numbers associated with the divisors of an integer*, by Mr. D. H. Lehmer. (Abstract No. 36-1-3.)

4. *A generalization of Lucas' functions U_n and V_n*, by Mr. D. H. Lehmer. (Abstract No. 36-1-4.)

5. *The semi-invariants and moments of incomplete normal and Type III frequency functions*, by Dr. C. C. Craig. (Abstract No. 36-1-5.)

6. *Sheppard's correction and semi-invariants*, by Mr. W. L. Langdon and Professor Oystein Ore. (Abstract No. 36-1-6.)

7. *The powers of a given rational prime which may divide exactly the discriminant of an algebraic field of nth degree*, by Mr. W. R. Thompson. (Abstract No. 36-1-7.)

8. *The derivation of tensors from tensor functions*, by Dr. A. K. Mitchell. (Abstract No. 36-1-8.)

9. *Expansion into infinite products of functions defined by Dirichlet series*, by Mr. Samuel Borofsky. (Abstract No. 36-1-9.)

10. *An application of the method of parameters to linear partial differential equations*, by Mr. F. H. Miller. (Abstract No. 36-1-10.)

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* The abstracts of papers will be printed in the future under a separate heading as explained in the announcement on pages 1–2 of this issue; those referred to in what follows are to be found on pages 45–64 of this issue. The cross-references to the abstracts will state the number of the volume and the number of the issue in which the abstract appears, as well as the serial number of the abstract. Thus, if an abstract numbered 238 is printed in issue No. 5 of volume 36, the cross-reference will be (Abstract No. 36-5-238).

11. *On a certain ring of functions of two variables*, by Professor J. F. Ritt. (Abstract No. 36-1-11.)

12. *Systems of linear partial differential equations*, by Professors Philip Franklin and C. L. E. Moore. (Abstract No. 36-1-12-*t*.)

13. *A lemma of the theory of differential systems*, by Professor J. D. Tamarkin. (Abstract No. 36-1-13-*t*.)

14. *Generalized indecomposable continua*, by Dr. P. M. Swingle. (Abstract No. 36-1-14-*t*.)

15. *Various forms of the fundamental functional in the problem of Plateau and its relation to the area functional*, by Dr. Jesse Douglas. (Abstract No. 36-1-15-*t*.)

16. *A general formulation of the problem of Plateau*, by Dr. Jesse Douglas. (Abstract No. 36-1-16-*t*.)

17. *Integral invariants and secular perturbations*, by Professor O. E. Glenn. (Abstract No. 36-1-17.)

18. *A formula in geometrical optics*, by Dr. T. H. Gronwall. (Abstract No. 36-1-18.)

19. *On Cayley's definition of non-euclidean geometry*, by Professor James Pierpont. (Abstract No. 36-1-19.)

20. *The ∞^3 plane sections of an arbitrary surface*, by Professor Edward Kasner. (Abstract No. 36-1-20.)

21. *On the Phragmén-Brouwer theorem*, by Professor W. A. Wilson. (Abstract No. 36-1-21.)

22. *A property of certain hyper-spaces*, by Professor W. A. Wilson. (Abstract No. 36-1-22-*t*.)

23. *On the representation of analytic functions of several variables as infinite products*, by Mr. J. M. Feld and Mr. Philip Newman. (Abstract No. 36-1-23.)

24. *A remark on a system of equations with infinitely many unknowns*, by Dr. L. W. Cohen. (Abstract No. 36-1-24.)

25. *Relations between the critical points and curves of a real analytical function of two independent variables*, by Dr. A. B. Brown. (Abstract No. 36-1-25.)

ARNOLD DRESDEN,
Associate Secretary

THE THANKSGIVING MEETING IN ANN ARBOR

The two hundred seventy-third regular meeting of the Society was held at the University of Michigan on Friday and Saturday, November 29–30, 1929. About ninety persons attended the meeting, among whom were the following sixty-three members of the Society:

B. E. Allen, W. L. Ayres, Barnett, Beale, G. A. Bliss, Blumberg, Bradshaw, Bushey, Churchill, Clack, A. H. Copeland, Copp, Dancer, H. T. Davis, Denton, Dushnik, P. D. Edwards, L. C. Emmons, Peter Field, Fisk, Folley, Fort, Gehman, J. W. Glover, L. M. Graves, Grennan, V. G. Grove, Hildebrandt, L. A. Hopkins, Ingraham, E. H. Johnson, L. S. Johnston, Karpinski, W. S. Kimball, Kuhn, Langford, Latimer, C. E. Love, McShane, Merriman, C. N. Moore, D. C. Morrow, A. L. Nelson, Nyswander, H. L. Olson, Parkinson, Plant, Pollard, V. C. Poor, Raiford, Rainich, Rasor, W. T. Reid, C. E. Rhodes, Ross, Roth, Running, Simmons, Speeker, Teach, J. H. Weaver, K. P. Williams, C. O. Williamson.

On Friday afternoon Professor Tomlinson Fort gave an address on *The general theory of factorial series*, and Professor L. M. Graves addressed the Society on *Discontinuous solutions in the calculus of variations*. Both of these addresses will appear in early issues of this Bulletin.

On Friday evening members and their guests attended a dinner at the Michigan Union. Professor Glover welcomed the Society to the University of Michigan and called on Professor Bliss to speak. Seventy-three attended the dinner.

Saturday morning a rising vote of thanks to the Department of Mathematics of the University of Michigan and others who had contributed to the entertainment of the Society was passed.

The titles and cross-references to the abstracts* of the papers read at this meeting follow below. Papers 1–7 were read on Friday morning, and papers 8–18 on Saturday morning. Paper 16, whose abstract number is accompanied by the letter *t*, was read by title. Mr. Abernethy was introduced by Professor Running, Dr. Baten and Mr. Chang by Professor Rainich, and Mr. Latshaw by Professor H. T. Davis. Professor Glover presided Friday morning; Professor Field, Friday afternoon; and Professors Bliss and C. N. Moore, Saturday morning.

1. *Exterior motion in the restricted problem of three bodies*, by Professor C. J. Coe. (Abstract No. 36-1-26.)

* See the footnote on page 6.

2. *Pairs of curves in an S_n*, by Professor G. A. Parkinson. (Abstract No. 36-1-27.)

3. *Centers of symmetry in analysis situs*, by Professor H. M. Gehman. (Abstract No. 36-1-28.)

4. *An investigation of surfaces in euclidean 4-space by means of 3-vectors*, by Mr. N. C. Fisk. Preliminary report. (Abstract No. 36-1-29.)

5. *Polynomial approximation by the method of least squares*, by Professor H. T. Davis and Mr. V. V. Latshaw. (Abstract No. 36-1-30.)

6. *A general explicit formula of numerical integration*, by Mr. J. R. Abernethy. (Abstract No. 36-1-31.)

7. *A calculus of variations problem whose extremals are parabolas*, by Professor H. A. Simmons. (Abstract No. 36-1-32.)

8. *The Hamilton-Jacobi theory for the problem of Lagrange in parametric form*, by Professor V. B. Teach. (Abstract No. 36-1-33.)

9. *Simultaneous treatment of discrete and continuous probabilities by use of Stieltjes integrals*, by Dr. W. D. Baten. (Abstract No. 36-1-34.)

10. *Gibbs' phenomenon in summable series of Bessel's functions*, by Professor C. N. Moore. (Abstract No. 36-1-35.)

11. *Some transformations of linear partial differential equations*, by Mr. H. C. Chang. (Abstract No. 36-1-36.)

12. *Note on an infinite system of linear differential equations*, by Dr. W. T. Reid. (Abstract No. 36-1-37.)

13. *On representation of integers by definite ternary quadratic forms whose determinant contains no square factors*, by Mr. A. E. Ross. (Abstract No. 36-1-38.)

14. *Nilpotent and idempotent algebras of infinite order*, by Professor M. H. Ingraham. (Abstract No. 36-1-39.)

15. *Residues of polygenic functions*, by Professor V. C. Poor. (Abstract No. 36-1-40.)

16. *Permutable functions in the complex domain*, by Dr. L. M. Blumenthal. (Abstract No. 36-1-41-*t*.)

17. *Sphere geometry and conformal transformations in function space*, by Professor I. A. Barnett. (Abstract No. 36-1-42.)

18. *The first variation of a triple integral in the case of variable limits*, by Professor H. A. Simmons. (Abstract No. 36-1-43.)

MARK H. INGRAHAM,
Associate Secretary

THE THANKSGIVING MEETING IN PASADENA

The two hundred seventy-fourth regular meeting of the Society was held on Friday, November 29, 1929, at the California Institute of Technology. Sessions were held in the morning and in the afternoon. The attendance included the following thirty-nine members of the Society:

O. W. Albert, Basoco, Harry Bateman, Clifford Bell, E. T. Bell, Bernstein, Birchby, Blichfeldt, Daus, Duncan, Elder, Engstrom, Garver, Glazier, Graesser, J. G. Hardy, M. W. Haskell, E. R. Hedrick, G. H. Hunt, Glenn James, B. W. Jones, L. S. Kennison, D. N. Lehmer, G. F. McEwen, W. E. Mason, Michal, Noble, T. M. Putnam, P. G. Robinson, Stager, Steed, Van Buskirk, Morgan Ward, Wear, Whiting, W. M. Whyburn, A. R. Williams, Wolfe, Worthington.

The meeting was presided over at different times by President E. R. Hedrick, Professor M. W. Haskell, and Professor E. T. Bell.

There was no meeting of the Council or of the Trustees of the Society.

A luncheon for members and their guests was held between sessions at the Maryland Hotel.

At the request of the Program Committee, Professor M. W. Haskell, of the University of California, delivered an address entitled *Autopolar configurations in the plane and in space.*

The titles and cross-references to the abstracts* of the papers read at this meeting follow below. The papers whose abstract numbers are accompanied by the letter *t* were read by title. Dr. Pall was introduced by Professor E. T. Bell, and Professor Uspensky and Mr. Ingram were introduced by Professor Hedrick.

1. *Autopolar configurations in the plane and in space*, by Professor M. W. Haskell. (Abstract No. 36-1-44.)

2. *Generalized Riccati differential equations*, by Professor W. M. Whyburn. (Abstract No. 36-1-45-*t*.)

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* See the footnote on page 6.

3. *On related difference and differential systems*, by Professor W. M. Whyburn. (Abstract No. 36-1-46.)

4. *Generalizations of Liouville's theorem and allied theorems*, by Professor E. R. Hedrick. (Abstract No. 36-1-47.)

5. *On certain quartic surfaces which are the envelopes of two systems of conicoids*, by Dr. A. R. Williams. (Abstract No. 36-1-48.)

6. *Three degeneracies in the theory of ternary quadratic forms*, by Professor E. T. Bell. (Abstract No. 36-1-49.)

7. *A triple of ternary quadratic forms*, by Professor E. T. Bell. (Abstract No. 36-1-50.)

8. *On certain quinary and senary quadratic forms*, by Professor E. T. Bell. (Abstract No. 36-1-51.)

9. *On Franklin magic squares*, by Professor D. N. Lehmer. (Abstract No. 36-1-52.)

10. *A system of integral equations implying and implied by a system of first order linear differential equations*, by Mr. W. H. Ingram. (Abstract No. 36-1-53.)

11. *Systems of integral equations and their Fredholm solutions*, by Mr. W. H. Ingram. (Abstract No. 36-1-54.)

12. *Application of the theory of integral equations to the theory of the electrical transmission line*, by Mr. W. H. Ingram. (Abstract No. 36-1-55.)

13. *On the transformation which reduces the Brioschi normal quintic to a general principal quintic*, by Professor Raymond Garver. (Abstract No. 36-1-56.)

14. *Concerning polynomial functions with certain properties*, by Professor Raymond Garver. (Abstract No. 36-1-57-*t*.)

15. *Newton's idea of God in the different editions of the Principia*, by Professor Florian Cajori. (Abstract No. 36-1-58-*t*.)

16. *On the representations of a constant times a square as the sum of an odd number of squares*, by Dr. Gordon Pall. (Abstract No. 36-1-59.)

17. *On certain irregular ternary forms with cross products*, by Dr. B. W. Jones. (Abstract No. 36-1-60.)

18. *In- and circumscribed sets of planes to space curves*, by Dr. Clifford Bell. (Abstract No. 36-1-61.)

19. *On the ideal theory of Zolotarev*, by Dr. H. T. Engstrom. (Abstract No. 36-1-62.)

20. *Fourier expansions of doubly periodic functions of the third kind*, by Dr. J. D. Elder. (Abstract No. 36-1-63.)

21. *The characteristic number of a sequence of integers satisfying a linear recursion relation*, by Dr. Morgan Ward. (Abstract No. 36-1-64.)

22. *Note on a table of linear forms*, by Professor P. H. Daus. (Abstract No. 36-1-65.)

23. *A simple proof of the theorem on the limit of probability*, by Professor J. V. Uspensky. (Abstract No. 36-1-66.)

B. A. BERNSTEIN,
Associate Secretary

BATCHELDER ON DIFFERENCE EQUATIONS

An Introduction to Linear Difference Equations. By Paul M. Batchelder. Published with the cooperation of the National Research Council. Cambridge, Harvard University Press, 1927. 209 pp.

The subject of difference equations is but one of the divisions of the rather extensive field of the calculus of finite differences. The other usually recognized divisions are interpolation, summation of series, and mechanical quadrature, and these topics, together with difference equations, were treated in the classic book by Boole, quite as if they formed an inseparable whole. But within recent years difference equations have received new attention, and the object of the later studies, and the whole spirit that pervades them, are so different from that of the old, that one is not surprised to observe a tendency to separate the larger subject. Such a separation is usually the characteristic of a growing study.

The work of Boole was what is customarily described as largely formal, and such a quality seems inherent in interpolation and mechanical quadrature. The revolt against pure formalism that has come with the function theory may have led some mathematicians to regard the virtues of these subjects as inferior ones, and their connection with calculation might tend to increase the disfavor in the minds of some. Such persons should experience some of the pleasure of the reformer at the rescue of difference equations from its old associates.

Though the subject of difference equations has been redeemed and has taken on the manners of the modern mathematical world, it has not enjoyed even a season of popularity. It may have been welcomed but it has been given no enthusiastic acclaim, for it has won few intimates and scarcely more acquaintances.

There are certainly merits in the new study, that perhaps may be described best by referring to the gamma function, which has long been so important. It was one of the first functions to be defined by a definite integral, and if it had done nothing more than generalize for positive reals the factorial, it would have been significant. But with the advent of the complex variable the function became more important, for the integral of Euler has meaning everywhere to the right of the axis of imaginaries, and the relation $f(x+1)=xf(x)$ continues the function to the left and defines it except at the points $0, -1, -2, \cdots$, where there are poles. Finally, by using a contour integral, Heine gave a representation uniformly valid where the function has a meaning. The behavior of the function in the distant parts of the plane is very notable, and the series of Stirling was among the first, and is still perhaps the best known, asymptotic series. It would of course be only natural to believe that the origin of all the interesting and peculiar analytic properties of the function lay in the simple difference equation $f(x+1)=xf(x)$ which it satisfies. But Hölder's proof that the gamma function satisfies no algebraic differential equation gave certainty to this belief, and helped to emphasize the importance of a general study of difference equations. As has been pointed

out in several places the first distinct contributions were by Nörlund, Galbrun, Carmichael, and Birkhoff, each using a different method.

One cannot help inquiring as to why the field has remained one of restricted interest. It may be due in part to the fact that as soon as one starts to generalize the problems become complex, and one cannot go far without the expense of great pains. This is conspicuous for instance in the recent study of certain "irregular cases" by C. R. Adams, who begins by giving series that formally satisfy the equation, and adds that such a property "can be verified directly, although the labor involved is not inconsiderable."* Such a remark will not cause a person familiar with the field to turn aside, but it is not likely on the other hand that it will make such appeal to the casual eye as will arouse either interest or curiosity. It is not implied by these remarks that difficulty is a hindrance to mathematical study, or that only the simple can be popular, for difficulty has always been a stimulus to study. But if a subject is to enjoy general favor it must give something more than the satisfaction that comes from overcoming formidable obstacles; there must be some other reward than that; there must be results with charm and a compelling interest of their own, or results that, perhaps unexpectedly, throw light upon another problem. And difficulties too must occur mingled with what is easy, for the simple but significant is not only refreshing, but it is for it that one makes the endless search. There is in the combination of what is difficult and what is easy, something that piques one, and haunts the mind with the thought that even the apparent difficulty may be only a stubborn mask. The real triumph does not lie in the mere discovery of truth, but in making it seem simple and natural, so that one feels comfortable and at home in its presence. Up to the present time it seems that the subject of difference equations has lacked such qualities, as well as any profusion of special problems that would mitigate the severity of the necessary but rather cold existence theorems.

The book by Batchelder seeks to introduce the reader to the subject of difference equations without bewildering him with all the intricacies of the general case. After certain preliminaries are disposed of, which include a consideration of asymptotic representation, the book treats the first-order equation with some completeness, the gamma function appearing as a special case. The hypergeometric equation is taken as a sufficiently representative equation of higher order, and one particularly interesting on account of the analogies with the well known differential equation. The development is along the line of Birkhoff's theory. An extensive treatment is given of the so-called irregular cases. Such cases had not been treated before and this part of the book constitutes the writer's own contribution. Even for the special equation the problem is involved.

It seems to the reviewer a fault to all but ignore the purely formal part of the subject. There are in the earlier part of the book some half dozen problems similar to the easier differential equations upon which one first practices. There could well have been a more liberal selection, for such problems not only serve to arouse the interest, but they contribute a great deal towards making one feel

* *On the irregular cases of the linear ordinary difference equation*, Transactions of this Society, vol. 30 (1928), pp. 507–541.

at home with new ideas, and when they are wisely chosen they can exhibit the variety that exists in a subject and suggest new lines of thought. Even simpler problems are not to be despised, for it is not a base thing to get pleasure from manipulating a problem into a practicable answer. One can never be sure that the most uncompromising devotee of existence theorems does not some times seek amusement in such indulgence.

All persons who have given any attention at all to difference equations must hope that the subject will find an appropriate place in the mathematical structure. It would seem that extensive treatises on analysis should devote some space to it, at least to the extent of treating the gamma function in a way different from the traditional one, and going a little beyond the gamma function. It is to be hoped that the material in Batchelder's book may help to make such treatments possible as well as stimulate further study of the subject.

K. P. Williams

HELLINGER-TOEPLITZ ON INTEGRAL EQUATIONS

Integralgleichungen und Gleichungen mit unendlichvielen Unbekannten. By E. Hellinger and O. Toeplitz. Reprint from the Encyclopädie der Mathematischen Wissenschaften with the addition of a preface by E. Hilb and of a special subject-index. Leipzig-Berlin, Teubner, 1928. Pp. 1335–1616.

In his preface Hilb points out that the book under review appears after a quarter of a century of research in the field of the theory of integral equations. It must be considered, therefore, as a survey of results obtained, and as an account of problems which remain still unsolved. "During several years of cooperative work the authors scrutinized the whole literature as to methods, results and their comparative range." We agree with Hilb that this "Report is indispensable to anybody who desires to penetrate deeply into this subject so extraordinarily important for its applications." An attentive reader, even well versed in the subject, will find many novel features in treating old and new questions, features which are extremely illuminating and inspiring; he will welcome the successful efforts of the authors to unify the multitude of existing methods and to present these methods as parts of a harmonious whole. As notable instances of this kind we may mention the treatment of completely continuous forms (pp. 1403–1413); of normal matrices (p. 1562; this seems to be a new notion introduced by the authors and proving to be quite useful in a number of recent investigations); of symmetrizable kernels (pp. 1536–1543) and matrices (pp. 1563–1575); of a general principle which could be designated as a "principle of preservation of the type" of kernels or matrices (pp. 1391–1392, 1431–1433). It is undoubtedly a good idea (although a departure from the usual style of the Encyklopädie) to give *proofs* of some facts of fundamental importance; the reader will be also pleasantly surprised to find references to some *facts not previously published* (Toeplitz, p. 1573; Toeplitz-Schmidt, p. 1575; Szász, p. 1522). It is hardly necessary to mention that the bibliography of the Report is extremely rich and shows that the authors have canvassed

almost everything hitherto published on the theory of integral equations.

All this being granted it seems desirable now to call attention to some points in the Report, which may present a certain interest.

1. The statement of any theorem consists of two parts of equal importance: (a) hypotheses, (b) conclusions. The balance between these two parts is not always preserved in the Report: references to some theorems are lacking in the exact formulation of the conditions of their validity and are, in this sense, somewhat misleading. As examples we may quote the references to St. Bóbr (p. 1446) who treated not the general case $p>1$ mentioned in the text, but only the particular case $p\geqq 2$; Mollerup (p. 1375); Lalesco (p. 1550) who proved the theorem of the text under restrictive conditions, while the proof of the general theorem has never been published, so far as we are aware; it is not clear what is meant on p. 1376 (top) by the statement that the resolvent and pseudo-resolvent are meromorphic functions of λ, the notion of the pseudo-resolvent being defined only for a discrete set of values of λ (characteristic values); there is an apparent contradiction in the references to General Analysis, on pp. 1475–1476: while the references on p. 1475 seem to indicate that there are results in General Analysis concerning and unifying the theory of bounded forms, the next page implies that Moore's hypotheses in Hilbert's case lead only to a case which is even more special than completely continuous forms.

2. The reader will find a more interesting example in this same subject in the treatment of general spaces on pp. 1446–1448. The authors raise here a very important question as to the possibility of extending Hilbert's theory of completely continuous forms to general spaces. Following an idea of Helly* they consider a space $(x)=(x_1,x_2,x_3,\cdots)$ of infinitely many dimensions, on a sub-space of which there is defined a "distance-function" $D(x_1,x_2,\cdots)$, satisfying the following postulates:

(i) $D(\lambda x_1, \lambda x_2, \cdots) = |\lambda|\, D(x_1, x_2, \cdots)$,

(ii) $D(x_1+y_1, x_2+y_2, \cdots) \leqq D(x_1, x_2, \cdots) + D(y_1, y_2, \cdots)$,

(iii) $D(x_1, x_2, \cdots) \geqq 0$; $D(x_1, x_2, \cdots) = 0$ implies $x_1 = x_2 = \cdots = 0$.

To these postulates they add a new one:

(iv) $$D(x_1, x_2, \cdots) = D(|x_1|, |x_2|, \cdots).$$

Simultaneously with the space (x) the authors consider the "polar" space (u), whose distance-function $\Delta(u_1, u_2, \cdots)$ is defined by the condition

$$\Delta(u_1, u_2, \cdots) = \text{upper limit} \left|\sum_{n=1}^{\infty} u_n x_n\right|, \text{ if } D(x_1, x_2, \cdots) \leqq 1,$$

provided the upper limit in question is finite. On page 1448 (footnote 224) it is stated that under the conditions indicated above the "Auswahlverfahren" and "Abspaltungsverfahren" (which calls for the use of Hilbert's theory of completely continuous forms) can be extended to the spaces (x), (u). As

**Über Systeme linearer Gleichungen mit unendlichvielen Unbekannten*, Monatshefte für Mathematik und Physik, vol. 31 (1921), pp. 60–91.

interesting as this observation may be, the following examples show that some additional postulates are necessary in order to make it correct.

Ex. 1. The "points" are all the continuous functions $x(t)$ defined on (0,1). The coordinates of the point x are the values assumed by $x(t)$ at all the rational points of (0,1). The distance-function $D(x_1, x_2, \cdots)$ is defined by

$$D(x_1, x_2, \cdots) = \int_0^1 |x(t)|\,dt.$$

It is now readily seen that, although the postulates (i-iv) are satisfied, there can be no question here of any "Auswahlverfahren" and, furthermore, the polar space (u) does not exist but reduces to the single point $(0,0,\cdots)$*.

Ex. 2. The space (x) consists of all the sequences $\{x_\nu\}$ for which $\sum_{\nu=1}^{\infty} |x_\nu|$ converges. The space (u) consists of all the sequences $\{u_\nu\}$ for which upper limit $|u_\nu|$ is finite. If the distance-function $D(x_1, x_2, \cdots)$ is defined by

$$D(x_1, x_2, \cdots) = \sum_{\nu=1}^{\infty} |x_\nu|,$$

it is readily seen† that the distance-function $\Delta(u_1, u_2, \cdots)$ is simply upper limit $|u_\nu|$. The postulates (i-iv) are satisfied again, but the form $\sum u_\nu x_\nu$ is not completely continuous (in the sense of p. 1401).

3. Although the bibliographical references of the Report as a rule are extremely complete, there are a few gaps, of which some, in the reviewer's opinion, are serious. Considerably more space (than half a page) should be given to the account of the fundamental work of Carleman, *Sur les équations intégrales singulières*, etc., which is the most important contribution to the theory of singular integral equations since the publications of Hilbert, Weyl, Hellinger, and Toeplitz. The significance of Carleman's work lies not only in the results obtained but also in the method used. Carleman, in distinction from Hellinger and Toeplitz, obtains his results without using the theory of forms in infinitely many variables; this procedure is particularly convenient in the general theory of non-bounded operators (compare the recent investigations of J. von Neumann and M. H. Stone). Carleman's thesis (*Über das Neumann-Poincarésche Problem für ein Gebiet mit Ecken*, Uppsala, 1916, iv+195 pp.) is not mentioned at all in the Report. It is true that the title of the thesis indicates its potential-theoretical tendency which is outside the scope of the Report. But it is also true that this thesis contains many results of importance in the general theory of integral equations, for instance a very interesting application of "Abspaltungsverfahren" to the case of an integrable kernel (under conditions more general than those used by Dixon, p. 1388, footnote 95); a treatment of symmetrizable kernels where many of the results of Mercer (p. 1543) are obtained a few years earlier than by Mercer; etc. The authors do not mention an important result due to Carleman (Acta Matematica, vol.

* This example was suggested to the reviewer by N. Wiener. Of course this example would be ruled out if the authors had formulated the property of *completeness* of the space (x).

† See Hahn, *Über Folgen linearer Operationen*, Monatshefte, vol. 32 (1922), pp. 1–88.

41(1918), pp. 377–384) that 2 is the exact value of the exponent of convergence of the sequence of the characteristic values of a nonrestricted continuous kernel. A wholly erroneous statement concerning the work of Carleman is to be found in the Report. On p. 1531, Footnote 453, it is stated that Carleman proved the *convergence* of the series $\sum(\lambda_\nu^+)^{1/(1-\alpha)}$ where λ_ν^+ are the positive characteristic values of the kernel $|s-t|^{-\alpha}H(s,t)$, $H(s,t)$ being continuous and $H(s,s)>0$ on a subinterval of (a,b), and $0<\alpha<1$. However, what Carleman actually proved is the *divergence* of the series $\sum(1/\lambda_\nu^+)^{1/(1-\alpha)}$! A. Pell's thesis should have been mentioned on p. 1495 before Lichtenstein's paper, in connection with certain integro-differential equations.

4. The attitude of the authors of the Report toward the methods of General Analysis is sufficiently clearly expressed on pp. 1471–1476 and 1595-1596. This attitude can be explained partly by the fact that, pending the publication of a comprehensive monograph devoted to this contribution of the American mathematical school, the most interesting results of General Analysis are inaccessible not only to foreign mathematicians but even to the majority of those in this country. Still we find statements like the one on p. 1596 too rash: since neither methods nor results existed for the general elementary divisors' theory of bilinear matrices, no new results could be obtained here by General Analysis (Für die Schule von E. H. Moore fehlte darum hier jeder Ansatspunkt).

5. Several passages scattered throughout the Report create an impression that the authors consider the facts (Lösungs*tatsachen*) as more important than methods of obtaining them (Lösungs*formeln*). This point of view is perfectly justified if it should mean that, of two methods leading to the same formal results, the preference should be given to the one which uses the minimum of assumptions. There are numerous cases, however, (and not only in applications) where the method used is the most important part of the problem. From this point of view it seems desirable that some particular problems (for instance, the moment problem, inversion of definite integrals, and reciprocal relations) should have been given more attention in the Report. An impartial reader may also find that the special method of *forms* in infinitely many variables occupies too much of the foreground of the picture in the Report, as compared with the theory of *integral equations* as such. It should be finally observed that applications are almost completely omitted from the scope of the Report, which was certainly the right thing to do; another article of the Encyklopädie devoted to the applications of the theory of integral equations is therefore an obvious necessity.

The defects mentioned above detract little from the great value and interest of the Report, which will for a long time occupy an important place on the desks of those who are interested in the growth of the immense structure of the theory of integral equations.

J. D. TAMARKIN

FRY ON PROBABILITY

Probability and its Engineering Uses. By T. C. Fry. New York, Van Nostrand, 1928. xiv+476 pp. $7.50.

This textbook, one of the series called *Books from Bell Telephone Laboratories*, is the outgrowth of a set of notes originally prepared for one of the courses of Bell Telephone Laboratories, and subsequently revised for use in a course of lectures delivered at the Massachusetts Institute of Technology during the second term of the year 1926–27. The first six chapters contain an introduction into the mathematical theory of probability, and the five last, applications to different subjects in statistics. In eleven appendices useful tables are given.

The mathematical chapters deal with permutations and combinations, elementary principles of the theory of probability, Bernoulli's and Bayes' theorems, distribution functions and continuous variables. Chapter VII, on averages, compares the mathematical averages such as "expectation," "expected deviation" to the averages taken as results of experiment, as "mean," "mean deviation." Then follow the normal law, Poisson's law, Pearson's curves, Gram - Charlier series, Pearson's criteria of goodness of fit, applications of the theory of probability to problems of congestion and fluctuation phenomena in physics, especially the Schottky effect. Correlation and the theory of small samples are not discussed in this book.

This survey of the contents shows how much useful information is contained in this new book on probability. It is especially written for technical students. The mathematics is carefully selected and explained so as to be understandable to such a public of readers, and many examples are taken from the field of technical applications. Bayes' theorem, for instance, is illustrated by this problem showing its usefulness in an interesting way.

"A factory produces a certain type of screw as a standard product. The screws are collected at the machine in boxes of 1200 each. Long experience has shown that the proportion of these boxes which contain various percentages of bad screws is substantially as follows:

Per cent of bad screws in the box	0	1	2	3	4	5	6
Proportion of boxes observed to contain this percentage of bad screws	0.78	0.17	0.034	0.009	0.005	0.002	0.000

Two per cent badness has been adopted as a manufacturing standard; that is, any box, which contains 2 per cent or less of bad screws is regarded as satisfactory, the aim of the inspection process being to reject those which are poorer. The normal inspection consists in the examination of 50 screws out of each box. A particular box, produced at a time when there was no special reason to suspect that the machines were not operating properly, showed 6 bad screws under

normal inspection. What is the probability that the manufacturing standard had not been maintained in the production of this box?"

This problem is to a certain extent typical for the book in its endeavour to connect good mathematics with technical applications. The chapter on problems of congestion, more than sixty pages, is the culmination point. There we are in the midst of the applications, mostly to telephone practice. Many mathematicians will here be astonished to see how many, how varied and how complicated, present day applications of the theory of probability to purely technical subjects are.

The author has also tried to vary the examples belonging to the ordinary theory of probability. Here he uses frequently variations of what he calls the "psychic research problem," in which a spiritualistic medium is supposed to be tested by an experiment with a certain number of black and red cards. The mathematical problem itself is simple enough. Continuous probabilities are illustrated with examples taken from the theory of gases. The Saint Petersburg problem has found its proper place in the chapter on averages.

Perhaps it is a pity that this "psychic research problem" has taken the place of some of the classical examples. De Moivre's problem, De Montmort's problem (or the problem of the drunken inhabitants of Chicago, in Coolidge's version), Fermat-Pascal's problem of the players, Bertrand's problem on the ruin of the players, cannot be omitted from a book on probability without a loss; the same holds for such a problem on continuous probabilities as the needle problem.

Sometimes there is an unsystematic arrangement of the material. Problem 8, p. 43, dealing with a test batch of lamps, leads to what Pearson has called the "hypergeometric law." This problem returns later as Ex. 26 in the psychic research problem. Then again as Ex. 28 is a "ball problem." Then at last the law is given as formula (25).

There is an excellent discussion of Bernoulli's theorem, with graphical representations that will make its meaning clear to all students. There is a clear distinction between the distribution curves dealing with the number of successes and those dealing with the ratio of the number of successes to the total number of events. Tchebycheff's generalizations are not discussed.

The treatment of continuous variables is not so good. There we have to meet the fact that the probability of hitting a set of zero measure in the continuum is zero. It has to be made clear that the theory of probabilities deals with a comparison of the measure of point sets and the corresponding theory of summation. These things are, of course, rather out of the way for the average technical student, but though the subject is touched, the idea of the set of measure zero is not clearly explained. In the same chapter we find an exposition of the change of variables in distribution functions of one and more variables, but the author does not point out clearly—as Bertrand and Poincaré do—that the choice of the variables is a matter of an a priori convention sometime urged upon us by the mechanics of the problem. We get other probabilities if we accept other variables, unless we change the weighting function.

The chapter on distribution functions contains not only an exposition of the normal law, but also of Poisson's law. The importance of this law has been

recognized more and more in recent years. The deduction from the binomial law by a curious limiting process for small probabilities does certainly not account for its great use. Borel, in the last edition of his book on probabilities, has given another, more satisfactory, deduction. Fry gives still another, more general than Borel's, basing his deduction on the two conceptions of "events happening individually at random" and "events happening collectively at random." He also deduces a generalization of Poisson's law on a slightly modified basis. This introduction to Poisson's law is, we believe, a very valuable part of the book.

Pearson's curves are not very extensively discussed; only a few types are mentioned. Gram-Charlier series are introduced after a short introduction to Hermitian polynomials. There Fry's exposition can easily be supplemented by Arne Fisher's book. This book is, also generally speaking, a very valuable supplement to Fry's book, for instance in the historical part. Fry makes very few remarks of a historical character, too few perhaps; but here Arne Fisher's book gives plenty of information.

Several problems on curve fitting are elaborately worked out, fitting by means of the binomial law, by Pearson curves, by the Poisson law as well as by Gram-Charlier series. An appendix with the principal statistics of these different curves is helpful in fitting problems. On p. 305 Fry admits, with Pearson, as a valuable representation, the "binomial" distribution

$$p(n) = C_n^{12.135}(0.339325)^n(0.660675)^{12.135-n}$$

with a broken exponent 12.135 instead of 12. It is doubtful whether or not this has much of a meaning.

One of the most valuable parts of the book is the tenth chapter, on problems of congestion. It is a systematic treatment of the distributions that may occur in the practise of telephoning.

The last chapter deduces the integro-differential equation of statistical mechanics for ideal gases, which yields the H function, the Boltzmann distribution and Maxwell's law of velocities. The book ends with some of the author's investigations on the Schottky effect.

A number of exercises, some very interesting ones, give the student an opportunity to test his knowledge. For technical students we strongly recommend the book, and students of more theoretical interests will find it instructive to see how many new practical applications the old theory of Pascal and Laplace has found.

D. J. Struik

HILBERT AND ACKERMANN ON MATHEMATICAL LOGIC

Grundzüge der theoretischen Logik. By D. Hilbert and W. Ackermann. Berlin, Julius Springer, 1928. 120 pp.

This book deals with mathematical logic very much after the fashion of the first volume of *Principia Mathematica*. The authors begin with a treatment of the theory of elementary functions of propositions, and go on, in their second chapter, to a consideration of the logic of classes and its application to the traditional syllogism; they then take up a discussion of general propositions that involve a use of the notions "some" and "all" as applied to variables denoting individuals; and finally, in their last chapter, they consider propositions involving generalization with respect to functions, in connection with which a discussion of the paradoxes, the theory of types, and the axiom of reducibility, is entailed. These are, of course, all well known topics; but there are certain features of the book that are peculiar to it, and it is to some of these features that we shall direct attention.

The authors begin their formal analysis of elementary functions (p. 22) with the four primitive propositions:

[a] $(p) : p \vee p \cdot \supset \cdot p,$

[b] $(p, q) : p \cdot \supset \cdot p \vee q,$

[c] $(p, q) : p \vee q \cdot \supset \cdot q \vee p,$

[d] $(p, q, r) : \cdot p \supset q \cdot \supset : r \vee p \cdot \supset \cdot r \vee q,$

which are identical with four of the five propositions employed in *Principia Mathematica*. They have also a rule of substitution and a rule of inference; but they omit one proposition used by Whitehead and Russell, namely $(p, q, r) : \cdot p \cdot \vee \cdot q \vee r : \supset : q \cdot \vee \cdot p \vee r$, because it can be shown to be a logical consequence of the remaining four. Now, a peculiarity of the way in which Hilbert and Ackermann deal with propositions [a]–[d] is this: they endeavor to show (pp. 29 ff.) that these propositions are, at once, *consistent*, *independent*, and *complete*, in the technical senses which these terms bear in connection with ordinary deductive systems. Their arguments in each of these cases call for comment, and we shall consider them in order.

In dealing with the question of consistency, the authors use an interpretational method, involving arithmetical products of 0 and 1. Of course, expressions [a]–[d] express propositions, and thus do not admit of interpretation as they stand; so that we must first abstract from the particular meanings of the symbols in question, and then re-interpret these symbols arithmetically. We need not concern ourselves here with the details of this argument; it involves showing that expressions [a]–[d], when given the arithmetical interpretation in question, are such that they, together with all the arithmetical products that can be derived from them by means of the two rules of deduction, have the value 0; a one-to-one correspondence is assumed to hold between these arithmetical propositions and the symbols, and again between the symbols and the original logical propositions; we are then told that if the original propositions could lead to a contradiction, some arithmetical product derivable from the primitive expressions would have to have the value 1.

It is somewhat surprising that the authors adopted such a roundabout procedure in endeavoring to establish a point that is immediately obvious; and the fact that they did do this leads us to suspect that the usual argument for consistency is not entirely clear to them. This usual argument, a form of which the authors adopt, is always one from *truth* to *consistency*; we know that no set of true propositions can be inconsistent, so that if we know that all the propositions of a set are true, we know that they cannot lead to contradiction. When we are concerned with a set of propositional functions, as in the ordinary case, we first assign values to the variables involved, and thus obtain a set of propositions, whose truth we may be able to observe; but, of course, when we are concerned from the outset with propositions, this first step, involving an interpretation, drops out. All we have to do, therefore, in order to see that propositions [a]–[d], together with the two rules, are consistent, is simply to observe that they are all plainly true. Hilbert and Ackermann seem to prefer to argue from the truth of certain arithmetical propositions to the consistency of the logical propositions with which they are concerned; but it is surely more to the point to argue directly from the obvious truth of these logical propositions.

In dealing with the question of independence, the authors employ interpretations similar to the one used in connection with consistency; they endeavor to show that no one of their primitive propositions follows from the remaining three. There is no occasion for examining this argument directly; it is quite certainly mistaken, for the sufficient reason that, in point of fact, [c] follows from [a], [b], and [d]. In order to see that [c] does follow, we may observe that a proposition of the form $p \vee q$ means simply that at least one of the two propositions in question is true, and that this is exactly what a proposition of the form $q \vee p$ means; so that these two symbols are not relevantly different. That is to say, the order in which p and q occur is not a symbolically significant feature of the symbol, just as the color of the ink is not, or the fact that the symbol is in italics. It is true that $q \vee p$ might, in some usage, be employed in a sense different from that assigned to $p \vee q$; but this cannot be so in connection with propositions, since only one meaning occurs, namely, "At least one of the two propositions is true"; there is simply nothing other than what is meant by $p \vee q$ left for $q \vee p$ to mean. But, now, since this is so, $p \vee q \cdot \supset \cdot q \vee p$ is merely a form of $p \supset p$; so that $(p, q) : p \vee q \cdot \supset \cdot q \vee p$ follows from $(p) \cdot p \supset p$; and this latter proposition is a consequence of [a], [b], and [d], as the authors show (p. 24).

With regard to completeness, two definitions are given, and an attempt is made to show that propositions [a]–[d] satisfy each of them. According to the first, a set of propositions is complete if, and only if, all true propositions of a certain domain can be derived from the set; and we are told that this means, in connection with the particular set in question, that all true universal propositions about functions of the sort occurring in the axioms can be derived. However, this restriction to universal propositions seems to be a mistake; for it seems clear that this definition ought to be understood to include true contradictories of universal propositions, and ought to mean that all these (that is, all that can be constructed in terms of the primitive notions employed) can be derived by means of the rules. If it does mean this, then the

assumptions would seem to be not complete in this first sense; for such true propositions as $\backsim(p, q) \cdot p \vee q$ can be formulated in terms of the primitive notions, and yet appear to be not derivable by the rules given. According to the second definition, a set is complete if, and only if, any proposition belonging to the domain in question is either implied by the primitive propositions or else incompatible with them. The authors establish the fact that this property holds by pointing out that all universal propositions of a certain kind follow from the axioms, and then showing that if other universal propositions followed, a contradiction would arise. Here, again, no explicit account is taken of contradictories of universal propositions; but it is easy enough to see that the argument applies to them. Thus, $(p, q) \cdot p \vee q$ is incompatible with the primitive propositions, so that $\backsim(p, q) \cdot p \vee q$ does actually follow, despite the circumstance that the two rules seem to be not sufficient for effecting its deduction.

In order to be able to deal with propositions involving generalization with respect to variables denoting individuals, the authors add two further primitive propositions to their list (p. 53); namely, $(x) \cdot f(x) \cdot \supset \cdot f(y)$ and $f(y) \cdot \supset \cdot (\exists x) \cdot f(x)$, where $f(y)$ means, according to their usage, that f holds for any value of y. It may be said, with regard to the first of these propositions, that $f(y)$ means precisely what is meant by $(x) \cdot f(x)$, and that therefore the proposition here in question is an immediate consequence of $(p) \cdot p \supset p$, and is thus superfluous. There is really no significant difference in logic between "any value" and "every value," these alternative expressions being employed for psychological reasons merely; so that if we wish to use $f(y)$ in addition to $(x) \cdot f(x)$, we are free to introduce it through a purely verbal definition. All that is required, then, is the second proposition, which becomes $(x) \cdot f(x) \cdot \supset \cdot (\exists x) \cdot f(x)$. Here, however, a criticism of another sort is relevant: in contrast to the other assumptions, this second proposition does not express a purely logical truth. This is so because $(x) \cdot f(x)$, being universal, is negative, and would be true if there were not at least one value within the range of significance of x, whereas $(\exists x) \cdot f(x)$ would be false; so that although in point of fact the proposition in question is true, there is no reason in logic why it should be. It is really very doubtful whether propositions like this second one ought to be admitted into a purely logical analysis at all, except as subordinate constituents of more complex expressions.*

In their final chapter, the authors deal, as we have said, with certain paradoxes that appear when we attempt to apply generalization to functions. They discuss, in the first place, the paradox that arises in connection with the notion of a predicate as being predicable of itself. Let $f(t)$ mean "t is predicable of itself," so that $\backsim f(t)$ will mean that t is not so predicable; and then consider the expressions $f(\backsim f)$ and $\backsim f(\backsim f)$, with regard to which, it is easy to see that if they were significant, they would be logically equivalent, so that a logical vicious circle would arise. The authors consider, in the next place, the paradox that appears in connection with expressions like: "This proposition is false," or "I am now asserting a false proposition." Here two situations arise, which

* For a discussion of the relations of universal to particular propositions, see Mind, vol. 36 (1927), p. 342.

they do not distinguish. The first of these expressions is most naturally interpreted as a proposition, p, which asserts, "p is false"; so that we have a proposition directly about itself, in connection with which a genuine vicious circle arises. But the authors do not consider this case; instead, they take the most natural interpretation of the second of the foregoing expressions, according to which it is to be rendered: "There is one and only one proposition which I am asserting, and it is false." Clearly, this is often true; but it cannot be true if I assert it, nor can it be false; and yet a vicious circle does not arise, since all that follows is that I never do assert it. Nevertheless, the situation here in question is just as objectionable as the occurrence of a vicious circle, since it leads to the conclusion that a logically possible state of affairs is in fact impossible.

C. H. LANGFORD

NÖRLUND ON FINITE DIFFERENCES

Leçons sur les Équations Linéaires aux Différences Finies. By N. E. Nörlund. Paris, Gauthier-Villars, 1929. vi+153 pp.

The theory of finite difference equations arose in investigations by Lagrange and Laplace; but it is only recently that the properties of the solutions of these equations have been developed with any considerable detail. The modern researches on this subject have been inaugurated by Poincaré and Pincherle. One owes to Poincaré a remarkable theorem on the manner in which the solutions of a linear homogeneous equation in finite differences behave for very large values of the variable. This theorem has been the point of departure of several investigations. In his preface the author says: "In recent years the theory of finite difference equations has been developed by a large number of authors, among whom may be mentioned G. D. Birkhoff, H. Galbrun, E. Hilb, E. Bortolotti, O. Perron, R. D. Carmichael, J. Horn, K. P. Williams, A. Guldberg and G. Wallenberg. [To this list, of course, should be added the name of Nörlund himself.] The subject is too vast for it to be possible to give here an exposition of all the results obtained. The aim of this book is to put in evidence the essential properties of the solutions of linear homogeneous equations, on the one hand by aid of their development in factorial series, on the other hand by aid of certain methods of successive approximations due to G. D. Birkhoff and R. D. Carmichael." Chapters I–IV are devoted to a single linear equation, different hypotheses relative to the coefficients being made in the different chapters. Chapters V–VI treat similar problems for a system of linear equations.

The first chapter is devoted to general properties of linear equations, such as the existence theorems which are readily proved, adjoint equations, equations with second member, and the reduction of the order of an equation by means of known solutions. A large part of this chapter is elementary and is devoted to the preliminaries of the general theory. There is, however, an existence theorem of considerable importance, based on hypotheses of a broad general character. In the fifth chapter one finds a treatment of precisely similar ques-

tions for a system of linear equations, each of the first order. The first and fifth chapters serve, therefore, to give the preliminaries of the theory, the one for a single equation, the other for a system of equations.

The second chapter is devoted to the solution of a single linear equation by means of factorial series. The necessary theory of factorial series is not developed in this book. It is supposed that the reader is already acquainted with the essential properties of these series. The definitions and results employed in the present exposition have been set forth by Nörlund himself in his book (Paris, 1926) entitled *Leçons sur les Séries d'Interpolation*, pp. 170–227. In many respects the theory developed in Chapter II is analogous to the classic theory of Fuchs for linear differential equations. Several of the artifices due to Fuchs and Frobenius in this latter theory have a similar application in the theory of difference equations. In some respects the existence theorems developed in this chapter are the most pleasing theorems in the whole field of the difference calculus.

The third chapter is devoted to an application of the transformation of Laplace to linear equations whose coefficients are polynomials. One of the tools required in the investigation is that of factorial series, so that this chapter has intimate contacts with the previous one.

On a first reading it is possible to omit Chapters II and III and pass immediately from Chapter I to Chapters IV, V and VI; and this can be done without destroying the continuity of thought. This fact may be useful to one who desires a knowledge of the main properties of the solutions of linear difference equations and who is unacquainted with the theory of factorial series.

In Chapter VI the solution of a linear equation by successive approximations is treated. The form of the method is that due to Carmichael and the detailed development follows closely his memoir of 1916 in the American Journal of Mathematics. The method is closely similar to and is an extension of the method already employed by the same author in the Transactions of this Society in 1911.

The final Chapter VI contains an exposition of the method of Birkhoff, principally as developed by him in 1911.

This book, though brief, contains the best-balanced introductory exposition of the theory of difference equations which has yet been prepared. The account is too short for extended details, but it is sufficiently comprehensive to give the reader an introduction to the various aspects of the theory of linear difference equations. A learner who is approaching this subject for the first time and who is equipped with the customary tools of modern analysis will find this book a convenient one to read early in his study of the subject. He might perhaps precede it by the reading of a portion of the still briefer book by Batchelder. He would do well to follow this present book by a study of the much more comprehensive treatise by Nörlund. With these three books before him, the student is well supplied with what is necessary for an introductory acquaintance with this modern subject of analysis. After a suitable study of these books he would naturally go then to the original memoirs.

R. D. CARMICHAEL

SHORTER NOTICES

Einleitung in die Mengenlehre. By A. Fraenkel. 3d ed. Berlin, Springer, 1928. 14+424 pp.

Mengenlehre. By E. Kamke. (Sammlung Göschen, No. 999.) Berlin and Leipzig, de Gruyter, 1928. 159 pp.

Books on the theory of aggregates all seem to present much the same approach to the subject. They begin with Cantor's definition of an aggregate and then build up a theory of cardinal and ordinal numbers. It seems to the reviewer that one of the main features of the aggregate theory is its extremely careful avoidance of any reasoning which is not clear cut and rigorous. For a student entering for the first time on rigorous thinking, the aggregate theory seems almost uselessly careful. But should a student have his first dip into rigor in this field? As a matter of fact most students get an introduction to rigor in a course in real variable. Then why, in a course on aggregate theory; and why, in the texts on the subject, is it not possible to put down a system of axioms which is considered sufficient for the purpose in hand and to proceed by indisputable steps to the erection of the structure of the theory? The two books about to be reviewed do not do this. The first, by Fraenkel, discusses the axioms very carefully, but not until the very end of the book! The second even proves the well ordering theorem without any mention of Zermelo's Axiom.

The treatise by Fraenkel on the theory of aggregates is now one of the finest. The book has been made much clearer by the definite statements of theorems in italics, which was not done in the previous editions.

This book is divided into five chapters: I. Foundations and Cardinal Numbers, II. Operations with Cardinal Numbers, III. Order Types and Ordinal Numbers, IV. Attacks on the Foundations and their Consequences, V. The Axiomatic Structure of the Theory of Aggregates and the Axiomatic Method.

The first three chapters cover the usual ground quite clearly except that what is assumed is sometimes a bit in doubt because no axioms are presented. The book contains problems which make it very suitable as a text for graduate students.

The fourth chapter has been enlarged to twice its former size and presents a most interesting discussion of Brouwer's new work and of Russell's work.

The fifth chapter discusses a system of axioms and Hilbert's theory of logic and metamathematics.

Not as much attention is paid in the book to the application of the theory to function theory or point sets as might be in a text of its size; it is a text on the theoretical side of the subject only.

In matters of printing and arrangement both author and publisher are to be complimented.

Kamke's little book is a very convenient summary of the subject of abstract aggregate theory. Its four chapters are concerned with, first, the generalities of the subject; second, cardinal numbers; third, ordered sets and ordered types; fourth, well ordered sets and their ordinal numbers.

Applications of the theory to point sets and function theory are sometimes given as examples to illustrate the matters discussed, but no detailed discussion of point set theory or of function theory is given. This is, of course, not to be expected in so small a book.

There is a short but very interesting discussion of the paradoxes.

It is to be hoped these other topics relating to the subject of this book will be dealt with in other volumes of the series.

T. C. BENTON

Höhere Mathematik für Mathematiker, Physiker, und Ingenieure. By R. Rothe. Part 2. Leipzig and Berlin, B. G. Teubner, 1929. viii+201 pp.

This little volume is the second of a series of three volumes intended to give the principal elementary facts of higher mathematics, for the benefit of mathematicians, physicists, and engineers. The present volume covers the integral calculus, infinite series and a brief treatment of vector analysis. The integration sections discuss formal indefinite integration processes, the definite integral, approximate integration by computation and mechanical means, and the definite integral as a function of a parameter. The infinite series section takes up the principal theory of convergence, a treatment of power series, expansions of special functions, and some applications. The last chapter is devoted to determinants, and the algebra of vectors, with applications to geometry and mechanics, mostly the former. On the whole one gets the impression that the physicist and engineer has had but scant attention in the brief mention of an occasional application to physics and mechanics. For instance, in integration second moments are not mentioned at all, and first moments only incidentally as mean value integrals.

The book has a number of interesting points and formulas. By way of example, there is developed an integration by parts formula of the nth order (suggestive of existence proofs in differential and integral equations); there are derivations of Wallis' product, Stirling's formula, and Euler's constant; determinants are defined by means of an expansion formula in terms of minors, which with the aid of mathematical induction can be (but is not by the author) made the basis for derivation of the principal properties of determinants.

The book makes an attempt and succeeds quite well in maintaining a high standard of rigor. There are occasional slips. For instance, in discussing the method of determining constants in dividing a rational fraction into partial fractions, the impression is given that the method of substitution for the variable is applicable only when the factors of the denominator of the given fractions are distinct; in stating Darboux's theorem on upper and lower integrals one is led to believe that the proof is an immediate consequence of earlier theorems, which it is not; the vector product of two vectors is expressed as a determinant, involving vector quantities, though determinants have been defined only for numbers.

On the whole, the book contains much that is worth while for the embryo mathematician, and presents the material in a very acceptable way.

T. H. HILDEBRANDT

The Theory of Determinants, Matrices, and Invariants. By H. W. Turnbull. London and Glasgow, Blackie, 1928. 8 vo, vii+338 pp.

Mechanical Features. This volume is attractively bound in smooth dark green cloth over extra heavy boards. It is printed from new and artistic type on special quality glazed paper and represents a real achievement in scientific printing on account both of the elegant arrangement of the formulas on the page and of the perfection of typography. The concise German symbolical notation for determinants and algebraic invariants, used consistently, afforded to the printers an opportunity to manufacture an exceptional mathematical book.

The Book as a Text Book. Scope. The reader of the work will do well to be reminded of the full title, as otherwise the impression might be gained that extensive study of the theory of determinants is necessary before invariant theory can be undertaken. This is hardly true literally.

We think that the author of this admirable book was not concerned primarily to produce a text-book. He probably was more interested in a scientific question; that of creating a logically developed exposition of the portion of invariantive algebra which has its foundations largely in linearity and the linear properties of determinants and matrices. In carrying out this plan he was able to draw relevant material from a wide range of original sources, and has produced a work very likely to make a place for itself as a fundamental text. The book also is very stimulating reading for one who has read the original memoirs first.

In confining ourselves to linearity in invariant theory we necessarily omit practically all of reduction to fundamental systems, and syzygetic dependence. There is naturally, however, a chapter on Hilbert's Lemma on the basis of a system of polynomials, and the consequent proof of Gordan's theorem. A compensating advantage inherent in the plan is that most linear properties, such as, for example, the Laplace identities, are immediately extensible by generalization from two or three variables or sets to n variables, the generalization of the binary theory in those respects where generalization is feasible. The extension of all theories to the consequences of a literal number of variables is a characteristic feature of Professor Turnbull's work.

There are twenty-one chapters, with titles as follows: Matrices and determinants. Fundamental properties of the determinant. Linear properties (and) fundamental Laplace identities. Multiplication of matrices and determinants. Linear equation (and) corresponding matrices. Special types of determinant. Differentiation of a determinant. Binary forms. The general linear transformation. General properties of invariants. The first fundamental theorem. Multilinear forms. Symbolic methods of reduction. Seminvariants (and) algebraically complete systems. The Gordan-Hilbert finiteness theorem. Clebsch's theorem. Applications of Clebsch's theorem, canonical forms, etc. Invariant equations and Gram's theorem. Geometric interpretations. The general quadric. Miscellaneous recent developments.

Numerous and excellent lists of examples are interspersed and essential references are included.

The Aronhold-Clebsch symbolism is both so expressive and so concise and the author possesses such skill in all that pertains to elegance of notation that this book gives to casual reading the impression of being more elementary than

it actually is. The author is persistent in giving the algorithms their simplest expression however. He uses the King's English happily and with simplicity and austerity.

Critical Evaluation. Allowing for the fact that a good book may, after the lapse of years, be superseded in whole or in part by another good book based upon the newer researches it remains true that a new work on invariantive algebra must justify its existence among the following competitors: Andoyer, *Leçons sur la Théorie des Formes*; Clebsch, *Theorie der Binären Formen*; Clebsch-Lindemann, *Vorlesungen ueber Geometrie*; Dickson, *Algebraic Invariants*; Elliott, *Algebra of Quantics*; Faá di Bruno, *Théorie des Formes Binaires*; Glenn, *Theory of Invariants*; Gordan, *Vorlesungen ueber Invariantentheorie*; Grace and Young, *Algebra of Invariants*; W. F. Meyer, *Allgemeine Formen und Invariantentheorie*; Salmon, *Lessons Introductory to Modern Higher Algebra*; Study, *Methoden zur Theorie der Ternären Formen*; Weitzenböck, *Invariantentheorie*; a book by Brioschi; and perhaps some others.

To be sure if we restrict our consideration to books in English, this book, due to its scope, would have few competitors if any.

The test of the newness of a new treatise is whether it offers a duplication of the master theorems presented in other books. If it does so it can have great value in the direction of popularization, scientific organization, pedagogy, etc., but not extensively as an original. We find in Turnbull's book a fairly novel layout of master theorems. This was possible because the subject, although only 125 years old, is vast and the number of books relatively small. Gordan planned another volume but did not compose it. Lindemann's enlarged edition of Clebsch's *Vorlesungen ueber Geometrie* was stopped by the war, and the second volume of W. F. Meyer's *Allgemeine Formen und Invariantentheorie* did not appear. As the author suggests in his preface, also, much of invariant theory in n variables is of comparatively recent origin.

In early chapters the elementary reducing identities of the binary symbolic theory are extended to the Laplace identities, the Sylvester identity and their generalizations. Parallel developments dealing with matrices, the simplex, tensors and dual transformations of the Sylvester identity (Chap. 4) lay the foundations for the theory of transformation and reduction in later chapters. Chapter 7 is an essay on Capelli operators, Cayley operators, jacobians and substitutional analysis. In Chapters 9 and 21 there are sections on orthogonality, linear transformation with absolute quadric, Hermite's theorem, and the affine group. One of the notable intervals in the book is Chapter 16 dealing with Clebsch's theorem on the expression of forms in terms of cogredient sets of compound variables, and the Gordan-Capelli series. Very remarkable is the recent Lasker-Wakeford theorem giving necessary and sufficient conditions that a given form be canonical. The complete concomitant system of the general quadric (in n variables x) and any number of linear forms, is developed in Chapter 20.

The author would rescue quantitative substitutional analysis from the position of aloofness which it occupies in the memoirs of Frobenius and Young. With this proposition the reviewer agrees, howbeit formalism and arithmetization are different questions in the general problem.

O. E. GLENN

Lehrbuch der Algebra. Volume 3: *Algebraische Zahlen.* By R. Fricke. Braunschweig, F. Vieweg und Sohn, 1928. vii+506 pp.

Each year there appears in Germany a new book on algebraic numbers. This fact reflects the central position of that subject in current research, as well as the richness of its material from which the various authors have made different selections.

This book devotes 189 pages to the general theory of algebraic numbers, 103 pages to cyclotomic fields (determined by a root of unity), quadratic fields and applications to forms $ax^2+bxy+cy^2$, and finally 211 pages to class invariants and class equations. The final topics relate to the complex multiplication of elliptic functions, with emphasis on the arithmetical and algebraic aspects, a subject first fully clarified by R. Fueter in his work in two volumes (Teubner, 1924, 1927). The pioneer book was Weber's *Elliptische Functionen und Algebraische Zahlen* of 1891, which was later counted as volume 3 of his *Algebra*. This explains the phrase on the title page of Fricke's book, "verfasst mit Benutzung von H. Weber's gleichnamigen Buche." While this was partially true of the first two volumes, it hardly applies to this third volume.

This text gives a readable introduction to the modern theory of algebraic numbers apart from the analytic theory of the distribution of primes and prime ideals. It makes use of O. Ore's recent simpler methods and extensions. In the final part of the book, Fricke was in his own speciality, the theory of elliptic modular functions.

The three volumes together make a worthy successor to Weber's *Algebra*, which is now out of print.

L. E. Dickson

Repertorium der höheren Mathematik. By E. Pascal. Second edition, completely remodeled, of the German translation, edited by E. Salkowski and H. E. Timerding. Volume I (*Analysis*), part 3. Leipzig-Berlin, Teubner, 1929. Pp. xii+1025–1598.

This is the final part of the *Analysis* volume of the second edition of the *Repertorium*; the first part appeared in 1910 and the second in 1927. It contains the following chapters: 20, modern theory of real functions, by E. Kamke; 21, modern developments in the theory of ordinary differential equations, by G. Hoheisel; 22, the theory of boundary problems in partial differential equations, by W. Sternberg; 23, finite differences, by A. Walther; 24, the theory of integral equations and functions of infinitely many variables, and their application to boundary problems in ordinary and partial differential equations, by H. Hahn, L. Lichtenstein and J. Lense; 25, trigonometric series, by A. Plessner; 26, spherical harmonics, Bessel and similar functions, by E. Hilb; 27, theory of numbers, by E. Bessel-Hagen. All of the authors have very successfully achieved their purpose of presenting concisely the main results of their subjects together with references to the more important literary sources, and the book will undoubtedly prove indispensable to the workers in any of the fields treated.

T. H. Gronwall

Vorlesungen über Natürliche Geometrie. By Ernesto Cesàro. German translation by Gerhard Kowalewski. Berlin, B. G. Teubner, 1926. Second edition. vi+352 pp.

This is a reprint of the first edition of the German translation published in 1901. The new feature is a valuable appendix by Kowalewski on *Generalized Natural Geometry.* Although extremely short, occupying only pages 332 to 342, this gives a clear and interesting account of the important theories founded by Georg Pick in 1906 and later developed by some of his students and also by Kowalewski.

Natural or intrinsic geometry, in its original form, as studied by Cesàro, deals with the differential invariants of the group of euclidean motions. The generalization made by Pick is to a very large category of continuous groups, including for example the projective and inversion groups. For each r-parameter group G_r in the plane (assuming a certain transitivity condition) a differential invariant of lowest order J is found (the analog of curvature), and also an integral invariant $s=\int \omega dx$ (the analog of arc length). The natural equation of a curve, relative to the given group, is then of the form $\phi(J, s)=0$. The curves defined by $J=$constant are the analogs of the W or anharmonic curves of projective geometry. Kowalewski has discussed the more general category $J=as+b$ in the Leipziger Berichte, 1924.

Pick's fundamental discovery of covariant coordinates is the final topic. The extension to space of three dimensions, which he has treated in his lectures, is still unpublished.

The terms *intrinsic* geometry and intrinsic equation, used by the Italian and English writers, seem more suggestive than *natural* geometry and natural equation, as used by the German school. The present reviewer introduced the term *natural* family of curves (Transactions, 1909, Princeton Colloquium Lectures, 1913) in another connection, namely in the theory of dynamical trajectories and optical rays. The simplest definition is the extremal families defined by $\int Fds=$ mimimum, where F is any point function and ds is the element of length. (Recently Schouten has called such families conform-geodesic.) It would be interesting to generalize this theory of natural families to other groups, ds being used then in the sense of Pick, and the reviewer expects to study this generalization.

EDWARD KASNER

Premières Leçons de Géométrie Analytique et de Géométrie Vectorielle à l'Usage des Élèves de la Classe de Mathématiques et des Candidats aux Grandes Écoles. By E. Lainé. Paris, Vuibert, 1929. 46 pp.

The students who begin their study of the subject with this little book will hardly find it too easy. The exposition is rather condensed and the number of exercises (23) is not sufficient to illustrate various topics treated in the text, of which some, as for instance, the theory of the central axis, perhaps should not belong to the "first lessons" on the subject. The book contains four chapters: Chapter 1. Scalars, vectors, coordinates. Chapter 2. Elements of vector calculus. Chapter 3. Analytic geometry, surfaces and lines, plane and straight line. Chapter 4. Systems of sliding vectors ("vecteurs glissants"). On the whole, it produces an impression of an outline rather than of a textbook.

J. D. TAMARKIN

Six Lectures on Recent Researches in the Theory of Fourier Series. By Ganesh Prasad. Calcutta, the University of Calcutta, 1928. xiv+139 pp.

This book gives an account of a series of lectures delivered at the University of Calcutta. It deals with recent developments in the theory of Fourier series, particularly those concerned with convergence, Cesàro summability, and properties of Fourier coefficients.

The first lecture is introductory in nature and outlines the various problems considered in the subsequent lectures. The second lecture deals with various criteria for the convergence of Fourier series, including those due to Lipschitz, Dini, Jordan, de la Vallée-Poussin, and Young. An account of Hardy's discussion of the logical relationship between these six criteria is then given and the lecture closes with a brief indication of Du Bois-Reymond's treatment of convergence in the case of discontinuities of the second kind and the consideration of two criteria for uniform convergence. The third lecture is concerned with examples of the failure of convergence and uniform convergence in the case of the Fourier series of continuous functions. It is based on the researches of Du Bois-Reymond, Schwarz, Fejér, Lebesgue, Steinhaus, and Neder.

The fourth lecture is devoted to the consideration of the Cesàro summability of Fourier series. It begins with the deduction of Fejér's criterion and Lebesgue's criterion for summability ($C1$). Following this the author discusses certain types of discontinuities of the second kind where Lebesgue's criterion fails to apply. He seems to regard this failure as a serious defect, whereas it is merely the logical result of the fact that the condition is not a necessary as well as a sufficient one. The lecture continues with various other criteria for summability; one due to the author to replace that of Lebesgue in the instances mentioned above, Lebesgue's criterion for summability ($C2$), Hardy and Littlewood's necessary and sufficient condition for Cesàro summability of some order, the Riesz-Chapman theorem regarding summability (Ck) where $k>0$, and a few other related results.

The fifth lecture deals with the notion of "strong summability," introduced by Hardy and Littlewood in 1913. After giving the original Hardy-Littlewood theorem and a simplified proof of it due to Fejér, the author proceeds to consider certain generalizations due to Carleman, Sutton, and Hardy and Littlewood themselves. The lecture closes with a criticism of these various criteria, which is of the same nature as his criticism of Lebesgue's criterion, referred to above.

The sixth lecture deals first with various properties of Fourier coefficients, the discussion being based on the researches of Riemann, Parseval, F. Riesz, Fischer, and Neder. Following this the author takes up certain questions connected with the uniqueness of the representation of a function by a trigonometric series, and discusses various results of this type due to Cantor, Bernstein, Menchoff, Zygmund, Rajchman, and Nina Bary. The lecture closes with an account of some researches concerning the relation between the Fourier series and the Legendre series of a function, due to Fejér, Haar, and Lukacs. An appendix deals with certain aspects of the so-called "Riemannian theory of trigonometric series," the discussion centering about the work of Riemann and Zygmund. A second appendix contains notes, additions, and corrections.

C. N. Moore

The Mathematical Theory of Relativity. By Th. De Donder. First edition. Cambridge, Massachusetts Institute of Technology, 1927. x+102 pp.

This little book contains the text of the lectures delivered by Professor Th. De Donder of the University of Brussels at Massachusetts Institute of Technology. They give an account of the main mathematical structure of the theory of relativity and some of the investigations of the author himself. First the "Minkowskian field," that is, special relativity, is introduced, and general point transformations are studied in this geometry. Then it is, mathematically speaking, easy to pass to the "Einstein field" of general relativity. The parallel displacement of Levi-Civita appears here as a formal consequence of the transformation theory. The study of kinematics leads to the covariant formulation of Coriolis' theorem, a special topic of study by Professor De Donder and some of his collaborators. The following paragraphs deal with the principal laws of point mechanics, electrodynamics of bodies at rest and moving, and the dynamics of continua; they are deduced from variational principles. In this respect the author has done much original research, and we find in this book a survey of what he has reached.

In a last chapter we find a relation of Schrödinger's wave equation to the theory of relativity; here we reach the covariant formulation of this equation given independently at about the same time by V. Fock and the author of this book. A generalization to continuous systems follows. This, however, is difficult to understand.

In a general introduction some of the outstanding points in the present state of the theory of relativity are discussed. As an example we mention the impossibility of accounting for positive and negative electricity, and the difficulties in the explanation of the experiments of D. C. Miller.

The book shows, as do all Professor De Donder's publications, his great mathematical ability. But in a publication like this we should have liked a broader introduction into the mathematical foundations of the theory of relativity. It is not emphasized at all that we have to do with groups and their theory of invariants and covariants. The contrast is the stronger if we compare this book with the recently published lectures of Felix Klein on the same subject. This is one of the reasons that the book will be hard to read for anyone not very familiar with the theory of tensors, and without some general physical and mathematical knowledge on the subject of relativity. The research worker, however, will find useful information in De Donder's book.

D. J. Struik

Congruenze Algebriche ed Esponenziali. Applicazioni. II. By Paolino Fulco. Civitavecchia, Moderno, 1928. 230 pp.

The book contains in extremely lengthy form the elements of the theory of higher congruences for a prime modulus, intended for a first course. Numerous numerical examples are inserted in the text. The author states in the introduction: "The major part of the results obtained are original, and are not to be found in the authors cited above, nor in others." I have, however, not been able to discover any new results of importance in the book.

Oystein Ore

Automorphic Functions. By Lester R. Ford. New York, McGraw-Hill, 1929. vii+333 pp.

This book will be welcomed by students and teachers of function theory. It is a distinct acquisiton to the text-book literature in analysis.

Workers in analysis know well how hard it has been to secure a respectable knowledge of the automorphic functions. To plod through Poincaré's memoirs, or the two-volume work of Klein-Fricke is hardly an appropriate task for a mathematician not specializing directly in the subject. Recent expositions of conformal mapping and uniformization, that in Bieberbach's *Lehrbuch*, for instance, give an easy approach to certain aspects of the theory, but a short and direct treatment of properly discontinuous groups, and the functions to which they lead, has hitherto been lacking.

In the first half of his book, Professor Ford gives a very simple and elegant treatment of groups of linear transformations, their fundamental regions and the functions invariant under the groups. The account given, while not exhaustive, is quite comprehensive. All that is presupposed of the reader is a knowledge of the elements of complex variable theory. It is interesting to notice that non-euclidean geometry, which figures conspicuously in all previous treatments of the subject, is not used at all.

Professor Ford employs a method of investigation developed by himself, which is based on the systematic use of the *isometric* circles of the linear transformations, that is, the circles at which the linear functions in the transformations have derivatives of modulus unity. This method seems to have had its origin in determinations made of the fundamental region of a properly discontinuous group by J. I. Hutchinson and by Humbert. The proofs assume a surprisingly simple form. This matter deserves special emphasis. Professor Ford's work, in the first part of his book, is not mere exposition. The methods which he creates are original, and of permanent scientific value.

The exposition is remarkably clear and explicit. A possible exception to this will be found in the discussion of the theta-series for groups in which infinity is a fixed point for certain transformations.

The second half of the book gives a detailed account of conformal mapping and uniformization, and considers some of the relations of automorphic functions to differential equations. These chapters can be read with great profit. As the author remarks himself, his treatment of conformal mapping is not as brief as those in which normal families of functions are used. In the uniformization chapter, the topological work is intuitive, rather than arithmetic.

An excellent second course in complex variables can be based upon this book. Teachers who have been following up first courses in complex variables by a semester of elliptic functions will perhaps welcome a chance to give a semester of automorphic functions instead.

The firm of McGraw-Hill is to be congratulated on having published this work. Together with the author, they have done students and teachers of analysis a brilliant service.

J. F. Ritt

Anwendung der Differential- und Integralrechnung auf Geometrie. Von Georg Scheffers. Dritte, verbesserte Auflage. Volume 1: *Einführung in die Theorie der Kurven in der Ebene und im Raume.* Berlin and Leipzig, de Gruyter, 1923. x+482 pp. Volume 2: *Einführung in die Theorie der Flächen.* Berlin and Leipzig, de Gruyter, 1922. xi+582 pp.

The first edition of this book (1901, 1902) was discussed in this Bulletin vol. 7 (1900–01), p. 144 and vol. 8 (1901–02), p. 332. In the years 1910–1913 a second edition was published, in which the author had corrected his text in many points, especially under the influence of Study, whose papers of 1908 and 1909 (Jahresbericht and Transactions) had criticized Scheffers' book and the ordinary treatment of differential geometry in general. This third edition has not many points of departure from the second. Scheffers' book remains one of the best textbooks on the subject; it deals with the material in a very pedagogic way and illustrates it with many interesting examples. Complex values of the coordinates are treated carefully. The point of view of invariants is emphasized.

It belongs to the "older" type of book on differential geometry, so that the modern reader misses not only an application of vector methods, but also the more recent developments of absolute differential calculus, the parallelism of Levi-Civita, and, last but not least, the differential geometry "im Grossen," the "macroscopic" differential geometry. The only truly modern book on the subject is Blaschke, *Differentialgeometrie.*

Scheffers' book contains many historical footnotes. These are sometimes very interesting. They give, however, only a set of separated facts about the history of the theories. A history of the development of differential geometry is not given in Scheffers' book; in fact, it does not exist at all. This only would give the proper background to the historical notes.

D. J. Struik

Quellen und Studien zur Geschichte der Mathematik. Edited by O. Neugebauer, J. Stenzel, O. Toeplitz. Part B: *Studien.* Volume 1, Section 1. Berlin, Springer, 1929. 1–112 pp. RM 12.

This is the first number of the first volume of source material and studies in the history of mathematics, brought out under the editorship of O. Neugebauer of Göttingen, J. Stenzel of Kiel, and O. Toeplitz of Bonn. The first of six articles in this number is by Toeplitz and deals in a speculative way with mathematics in the writings of Plato. He opposes the thesis of A. E. Taylor (*Mind*, vol. 35, pp. 419–440; vol. 36, pp. 12–33) that Plato had evolved the concept of irrational number substantially as set forth in modern times by G. Cantor. Toeplitz sets up a thesis of his own, that Plato had an important ratio-concept playing a fundamental role in his general theory of ideas. Another article, by F. Solmsen, stresses the influence of Plato in the evolution of the mathematical method—the method which draws necessary conclusions. Stenzel discusses the logos of Aristotle.

Of great interest is Neugebauer's discussion of recently discovered Sumerian and Babylonian tablets which disclose not only a very early use of sexagesimal numbers and sexagesimal fractions, but also a knowledge of geometry not hitherto found in ancient Babylonian records. The area of a trapezoid is given.

There appear also somewhat complicated relations involving the sides of similar right triangles in problems requiring the solution of quadratic equations like $x^2-16x-80=0$ and $3x^2-44x-320=0$. The only pre-Grecian solution of quadratic equations known up to the present time had been the Egyptian, in problems leading to the pure quadratic form $x^2=c$. The geometric figures on the Babylonian tablets are defective, suggesting to Neugebauer the remark that the characterization of geometry as the science which draws correct conclusions from incorrect figures applies to its very beginning. Another article prepared jointly by Neugebauer and W. Struve, of Leningrad, reveals that the Babylonians measured the length of the circle by $C=d\pi$ (d=diameter), and the area of a circle by $A=C^2/12$, where $\pi=3$. The rule for the circular area in the Egyptian Rhind papyrus involves a more accurate value of π, namely, $\pi=3.16$. New also is the conclusion that the Babylonians were familiar with the *theorem of Thales*, that a triangle inscribed in a semi-circle is a right triangle; and with the *Pythagorean theorem*, at least for the sides 20, 16, and 12. These theorems are used in the computation of the length of a chord of a circle. Another new historical find is the Babylonian computation of the volume of the frustrum of a cone, by multiplying the arithmetic mean of the upper and lower basal areas by the altitude. In marked contrast to this mere approximation is the Egyptian accurate computation of the volume of the frustrum of a square pyramid (*Ancient Egypt*, 1917, p. 100). Historical investigations of the present century indicate that ancient Babylonian and Egyptian mathematics was much more highly developed than was formerly supposed.

FLORIAN CAJORI

Vektoranalysis mit Anwendungen auf Physik und Technik. By R. Gans. 6th edition. Leipzig and Berlin, B. G. Teubner, 1929. viii+112 pp.

This excellent little book is too well known and appreciated (six editions in 24 years!) to need a detailed review. Suffice to say that a mathematician as well as physicist will find there unexpectedly rich material, which is selected with great care. The book should be highly recommended as an introduction or as a "first help" to students interested in applied mathematics. The value of the book would still increase if it were supplied with exercises, which, unfortunately, are entirely absent. This is a point to be improved upon in the next edition, which undoubtedly will appear before long.

J. D. TAMARKIN

Operational Circuit Analysis. By Vannevar Bush, with an appendix by Norbert Wiener. New York, John Wiley and Sons, 1929. x+392 pp.

This is an elementary text on the Heaviside theory, for engineers. The method of exposition is largely that of Carson; a pleasing innovation is the inclusion, in Chapter 13, of some of the points of view of Wiener's paper in Mathematische Annalen, volume 95. There is a large collection of problems and the presentation appears in general to be satisfactory from the standpoint of the engineer, although one might occasionally wish for somewhat more precision in the statement of results. In the appendix, Wiener gives a quite pleasing description of some of the high points in Fourier analysis.

T. H. GRONWALL

Introduction to Theoretical Physics. By Leigh Page. New York, Van Nostrand, 1928. x+587 pp.

This book, which is based on a course of five hours a week for a year, given by the author at Yale, covers in a very satisfactory but, necessarily, not very detailed way those parts of theoretical physics with which every graduate student should be acquainted. An introduction (39 pp.) gives the usual theorems of vector analysis and the vector notation is consistently used in the subsequent portions of the book. Part I (pp. 41–84) treats the dynamics of particles, of rigid bodies, deformable bodies and advanced dynamics (including a useful section on phase integrals and action variables). Part II (pp. 185–248) discusses the hydrodynamics of perfect and viscous fluids. Part III (pp. 249–316) is devoted to classical thermodynamics, statistical mechanics, and the kinetic theory of gases. Part IV (pp. 317–484) treats electrostatics and magnetostatics, electric currents and electromagnetic theory (including such topics as the principle of relativity, the Compton effect and the distribution of energy in the normal radiation spectrum). The concluding Part V treats of geometrical optics, physical optics, and the origin of spectra (including the Bohr theory of the hydrogen and helium spectra, Stark effect, electronic bands).

It is quite clear that any attempt to give an account of so many things in one book must forego any particular novelty of treatment and must have usefulness as its aim. The book under review should prove very useful for many years to students beginning graduate work and its careful exposition will permit many an instructor to treat in detail matters in which he is particularly interested. The printing is excellent and the proof reading must have been done with great care.

The following minor points seemed to call for some criticism. The definition of a vector on p. 1 is a tautology since no explanation of, or criterion for, what is meant by *having direction* is given. This is cared for subsequently in §8. Since the vector product $\mathbf{p}\times\mathbf{p}$ is the zero vector and not the scalar zero, the zero (p. 6) should be set in black type. The definitions of velocity and acceleration on p. 42 might well bring out clearly the fact that these terms are defined *relative to a given reference frame.* It is necessary to show that the angular velocity vector (as defined on p. 86) actually is a vector; the statement on p. 88 describing the analytical character of the difference between linear and angular velocity is hardly satisfactory, the real distinction being that what is called angular velocity is really a dyadic or linear vector function. The significance of the word rigid (p. 100, l. 15) is not clear. The definition of number of degrees of freedom on p. 168 is valid only for holonomic systems and the discussion of p. 169 is applicable, as given, only to systems having fixed constraints. The reason for fixing the limits of the phase integral for a cyclic coordinate (p. 181) as 0 and 2π is not explained.

We recommend this book to all mathematicians interested in applied mathematics (may their tribe increase). For this group of readers the work would be greatly improved if suggestions for further reading were given at the end of each chapter.

F. D. MURNAGHAN

A Debate on Relativity, with an introduction by William Lowe Bryan, President of Indiana University. *Favoring the theory*: R. D. Carmichael, H. T. Davis. *Opposing the theory*: W. D. MacMillan, M. E. Hufford. Chicago and London, Open Court Co., 1927. viii+154 pp.

This book is an account of a debate on the theory of relativity held at the University of Indiana on May 21 and 22, 1926. Professors Carmichael and MacMillan discussed the theoretical aspect of the theory, the two other debaters the experimental aspect. In this way the reader gets an opportunity to review the different ways of reasoning and finds at the same time a place where he can get some information about recent experiments relating to Einstein's theory. We find a discussion of Miller's experiments (1921-'25), those of Michelson and Gale (1925), of St. John (1923), of Tomaschek (1925). The repetition of Michelson's experiment in a rotating balloon by Picard and Stahel (1926) was too recent for mention.

The main argument of Professor Hufford against the conclusiveness of the experimental evidence is that the theory of relativity is not the only way to explain the different experimenta crucis. Against Michelson's experiment he sets Miller's results. He uses Poor's criticism of the way in which Campbell has interpreted the results on the bending of light rays during the sun eclipse of 1922. Such explanations are, however, always explanations ad hoc, as Professor Davis points out.

Professor Carmichael gives an interesting exposition of the outstanding elements of Einstein's theoretical work. He emphasises the simplicity of the theory, that is not in the least affected by the rather complicated mathematics required for its mastery. We might add here that the same criticism was heard by Newton against his exposition. Calculus was to many of Newton's contemporaries just as esoteric as tensor calculus is to many modern scientists. Even Huygens confessed that he could not master it sufficiently.

It is a pity that the theoretical aspect of the theory has so unsatisfactory an opponent as Professor MacMillan. Professor MacMillan has had the unhappy idea to base his opposition against the theory of relativity on so-called "normal intuitions." For him the euclidean character of physical space and the newtonian character of physical time are "normal intuitions." What these things are may be illustrated by another example given by Professor MacMillan.

"The physical universe is continuous in time. This postulate asserts that the universe had no beginning and will have no end. It is at this point that we part company with many of the theologians."

And with all mathematicians, I believe. Continuity is not defined by "having no beginning and no end." Among the other examples there are some that seem equally insufficient. "Normal intuitions" are not so normal after all.

If our conceptions about space and time are so vague, it is really time to criticise them severely. But Professor MacMillan even tells us that Newtonian mechanics, being based on these same normal intuitions, is beyond any doubt whatever. With these fixed ideas he cannot believe in Einstein's interpretations and he advises us to wait till we find a better way to account for the difficulties in the explanation of natural phenomena.

It is always interesting, when we face this kind of criticism of Einstein's

theory to recall how bitterly Newton's mechanics was attacked around 1700. It did not at all agree with the "normal intuitions" of the people of those days. Men like Swift, Berkeley, Leibniz protested against those freaks of absolute space and universal attraction. And it is exactly because of the fact that Newton's mechanics does violate some of the "normal intuitions" of the human mind in the last centuries—compare the criticism of Euler, Carl Neumann, Mach, of Newton's conception of rotation—that Einstein's theory was born.

D. J. STRUIK

Encyclopädie der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Volume 2 in three parts. *Analysis*, edited by H. Burkhardt (1896–1914), W. Wirtinger (1905–1912), R. Fricke and E. Hilb. Third part, second half. Leipzig, Teubner, 1923–1927. xiii+675–1648 pp.

Mathematicians who eagerly expected the publication of this volume certainly will not be disappointed; it is a result of careful and thorough work on the part of a number of leading specialists in their respective fields. The volume contains an enormous amount of bibliographical material which is presented in readily accessible form. It is difficult to imagine a mathematician engaged in research or even in teaching the subjects discussed, who will fail to benefit by repeated consultation of this volume.

The volume under review consists of four parts which were published separately at different dates.

Heft 6 (July 14, 1923) contains the articles by Nörlund, *Neuere Untersuchungen über Differenzengleichungen*; Bohr and Cramér, *Die neuere Entwickelungen der analytischen Zahlentheorie.*

Heft 7 (April 1, 1924), Borel and Rosenthal, *Neuere Untersuchungen über Funktionen reeler Veränderlichen*, contains three parts: Zoretti and Rosenthal, *Die Punktmengen*; Montel and Rosenthal, *Integration und Differentiation*; Fréchet and Rosenthal, *Funktionenfolgen.*

Heft 8 (September 10, 1924) contains the article by Hilb and M. Riesz, *Neuere Untersuchungen über trigonometrische Reihen*; Hilb and Szász, *Allgemeine Reihenentwickelungen*; Lichtenstein, *Neuere Entwickelung der Theorie partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus.*

Heft 9 (December 15, 1927), Hellinger and Toeplitz, *Integralgleichungen und Gleichungen mit unendlichvielen Unbekannten.*

A detailed analysis being entirely out of place here (we give a separate review of the last article by Hellinger and Toeplitz, which was published also as a separate book) let us point out only the desirability (*i*) of mentioning boundary problems and the notion of Green's function in the theory of difference equations, for example, M. Bôcher, *Boundary problems and Green's functions for linear differential and difference equations* (Annals of Mathematics, vol. 13(1911–12) p. 71–88). (ii) of giving a place to the theory of q-difference equations as a field closely related to that of difference equations. (iii) of having an author index at the end of the volume. The absence of such an index is unfortunately a general defect of the Encyklopädie.

J. D. TAMARKIN

NOTES

The Mathematical Association of America announces the publication of volume 2 of the *Rhind Mathematical Papyrus*, by A. B. Chace, L. Bull, and H. P. Manning, with a bibliography of Egyptian and Babylonian mathematics by R. C. Archibald, and a description of the mathematical leather roll in the British Museum, by S. R. K. Glanville. This volume contains the photographic fac-simile, the hieroglyphic transcription, the transliteration, and the literal translation. The work is published under the auspices of the Mathematical Association through the Open Court Publishing Company.

A new journal entitled Studia Mathematica has been founded at Lwow, under editorship of Stefan Banach and Hugo Steinhaus.

Professors Artur Haas, of the University of Vienna, and F. G. Donnan, of the University of London, have been invited by the Gibbs Committee (representing the departments of physics and chemistry of Yale University) to be editors of a commentary on the works of J. Willard Gibbs. A number of distinguished physicists and chemists will collaborate with them.

The Academy of Sciences of Amsterdam has awarded the Roozeboom gold medal for contributions to physical chemistry to Dr. J. J. van Laar, formerly of the University of Amsterdam.

The Nobel prize in physics for 1928 has has been awarded to Professor O. W. Richardson, of King's College, London. The prize in physics for 1929 has been awarded to the Duc de Broglie, of Paris.

The Verein Deutscher Ingenieure has awarded the Grashof medal to Professor L. Prandtl.

Cambridge University has awarded its John Winbolt prize to J. N. Goodier, of Downing College, and H. S. Sayles, of St. John's College, for their joint dissertation on *Some problems of plane stress.*

The Royal Society of London has awarded its Davy medal to Professor G. N. Lewis, of the University of California, for his contributions to classical thermodynamics and the theory of valence, its Copley medal to Professor Max Planck, of the University of Berlin, for his contributions to theoretical physics and especially as originator of the quantum theory, and a Royal medal to Professor J. E. Littlewood, of Cambridge, for his work in mathematical analysis and the theory of prime numbers.

The Technical School of Danzig has conferred an honorary doctorate on Professor C. W. Oseen, of Upsala.

The Saxon Academy of Sciences has elected as corresponding members Professors David Hilbert, of Göttingen, and Max Planck, of Berlin.

On the occasion of the opening of its new chemical laboratory, Princeton University conferred honorary doctorates on Irving Langmuir, of the General Electric Company, and Jean Baptiste Perrin, director of the laboratory of physical chemistry at the University of Paris.

On the occasion of its 175th anniversary, Columbia University conferred honorary doctorates on Dean H. E. Hawkes and Professors C. J. Keyser and D. E. Smith, among others.

Dr. Max Mason has been elected president of the Rockefeller Foundation.

Professor Edwin B. Wilson, of Harvard University, has been elected president of the Social Science Research Council.

During the past year, Dr. J. Douglas has presented his solution of the problem of Plateau in a series of several lectures before the seminars of Professors Hadamard at Paris (January, 1929), Courant and Herglotz at Göttingen (June, 1929), and Blaschke at Hamburg (July, 1929), and has been invited to present his further work on the problem in Professor Hadamard's seminar this winter. Dr. J. A. Shohat presented his work on the approximation of continuous functions and definite integrals in Professor Hadamard's seminar in December.

At the University of Dijon, Professor A. Boutaric has been appointed to the professorship of physics.

Professor P. Flamant, of the chair of rational mechanics at Clermont-Ferrand, has been appointed professor of general mathematics at the University of Strasbourg.

Professor Bela de Kerékjárto, of the University of Szeged, delivered a course of lectures on topological problems at the Institut Henri Poincaré during the first semester of 1929–30.

Dr. Erwin Fues has been appointed professor of applied mathematics at the Hannover Technical School.

Dr. Klose, of the University of Riga, has been appointed to an associate professorship at the University of Berlin.

Professor Wilhelm Müller, of Hannover, has been appointed professor of mechanics at the German Technical School at Prague.

Dr. Heinrich Schatz, of the University of Innsbruck, has been promoted to an associate professorship of mathematics.

Mr. R. V. Southwell, lecturer at Trinity College, Cambridge, has resigned.

Dr. F. R. Bamforth has been appointed assistant professor of mathematics at Cornell University.

Professor M. A. Brumbaugh, of the University of Pennsylvania, has been appointed professor of statistics in the school of business administration of the University of Buffalo.

Professor E. F. Cox, of West Virginia Collegiate Institute, has been appointed to an associate professorship at Howard University.

Assistant Professor D. L. Holl, of Iowa State College, has been promoted to an associate professorship.

Assistant Professor E. B. Lytle has been promoted to an associate professorship of the teaching of mathematics at the University of Illinois.

Mr. R. H. Marquis has been appointed assistant professor of mathematics at Ohio University, Athens.

Dr. G. M. Merriman has been appointed assistant professor of mathematics at the University of Cincinnati.

Mr. E. L. Mickelson has been appointed professor and head of the department of mathematics at New Mexico State Teachers College, Silver City.

Dr. Oystein Ore, of Yale University, has been promoted to a professorship of mathematics.

Dr. M. M. Slotnick, of Harvard, has been apppointed assistant professor of mathematics at Grinnell College.

Mr. C. A. Spicer has been appointed professor of mathematics at Western Maryland College.

Professor J. C. Tinner, of Wilberforce University, has been appointed to a professorship at Bishop College.

The following appointments to instructorships are announced:
University of British Columbia, Mr. F. J. Brand;
Cornell University, Mr. J. M. Clarkson;
Crane Junior College, Miss Edna M. Feltges;
Houston Junior College, Mr. W. A. Rees;
University of Iowa, Dr. J. M. Earl;
Massachusetts Institute of Technology, Mr. S. B. Littauer;
Princeton University, Dr. K. E. Rosinger (in the department of philosophy);
Rutgers University, Mr. H. B. Huntley;
Union College, Mr. A. H. Fox and Mr. F. S. Lerch;
State College of Washington, Mr. John Biggerstaff and Mr. L. G. Butler;
Yale University, Dr. A. K. Mitchell.

Professor A. Bindoni, of the Istituto Tecnico di Reggio Emilia, died December 17, 1928, at the age of forty-nine.

Dr. P. Fatou, of the Paris Observatory, died in September, 1929.

Dr. August Horstmann, professor emeritus of theoretical physical chemistry at the University of Heidelberg, died recently at the age of eighty-seven.

Auguste Lebeuf, director of the Observatory of Besançon, known for his work in chronometry and celestial mechanics, died at the age of seventy.

Professor Carlo Rosati, of the University of Pisa, died August 19, 1929, at the age of fifty-three.

Dr. Gottfried Rückle, of the Astronomisches Institut für kleine Planeten in Frankfurt, died in September, 1929. He was known as a phenomenal lightning calculator who also attained prominence in mathematics and astronomy.

Professor T. P. Trivedi, of the D. J. Sind College, Karachi, India, died in April, 1929. Professor Trivedi was a member of the American Mathematical Society.

Mr. Arthur Berry, vice-provost of Kings's College, Cambridge, died August 15, 1929, at the age of sixty-six. Mr. Berry had been a member of the American Mathematical Society since 1897.

Dr. T. J. I'a. Bromwich, formerly fellow and praelector in mathematical science at St. John's College, Cambridge, and University lecturer in mathematics, died August 24, 1929, at the age of fifty-four. Dr. Bromwich had been a member of the American Mathematical Society since 1902.

Professor C. C. Engberg, of the University of Nebraska, died September 21, 1929. Professor Engberg had been a member of the American Mathematical Society since 1898.

Professor C. R. MacInnes, of Princeton University, died September 29, 1929, at the age of fifty-three. Professor MacInnes had been a member of the American Mathematical Society since 1911.

Professor R. M. Mathews, of the University of West Virginia, died recently. Professor Mathews had been a member of the American Mathematical Society since 1910.

Miss Sarah E. Smith, professor emeritus of mathematics at Mount Holyoke College, died November 18, 1929.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THIS SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume.* Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.†

1. Dr. A. A. Albert: *The structure of pure Riemann matrices with non-commutative multiplication algebras.*

The chief outstanding problem in the theory of Riemann matrices is the determination of the structure of all pure Riemann matrices ω with a given non-commutative division algebra B as multiplication algebra D. The problem is here reduced essentially to the case where B is a normal division algebra. We consider in detail the case where B is a known normal division algebra, that is, an algebra in n^2 units and of type R_n. In that case we find not only the structure of ω but that of its principal matrix, and obtain necessary and sufficient conditions on D that it be a multiplication algebra of some ω, and the form of the ω in terms of a single arbitrary submatrix. The theory is applied to the cases of generalized quaternion algebras, generalized quaternion algebras over a quadratic field, and to Cecioni algebras. In the first two cases actual classes of numerical examples of Riemann matrices are given, these being the first known examples of Riemann matrices with non-commutative division multiplication algebras.

2. Dr. A. A. Albert: *A construction of rational cyclic division algebras of order sixteen.*

As is known, all normal division algebras in sixteen units are Cecioni algebras. However, the only algebras of order sixteen which have been constructed are the few isolated algebras of Cecioni. We consider here

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* In the future, abstracts of papers will appear under the heading given above, separately from the reports of meetings, as explained in the announcement of pages 1–2 of this issue. Eventually, many abstracts will be printed in advance of the meeting at which the corresponding paper is presented, but this will not be done until the abstracts of papers that have been read have been printed.

† Thus, if an abstract numbered 238 is printed in issue No. 5 of volume 36, the cross-reference will be (Abstract No. 36-5-238).

the class of algebras called cyclic (Dickson) algebras in sixteen units, a subset of the set of all decioni algebras. The known conditions that such algebras be division algebras are merely sufficient, and have not been shown to be necessary. We find necessary and sufficient conditions that such algebras be division algebras, and, by applying our theory to the case of algebras over the field of all rational numbers, construct an infinity of cyclic algebras for every cyclic quartic in the canonical form

$$\omega^4+2\nu(\Delta_1^2+\Delta_2^2)\omega^2+\nu^2\Delta_2^2(\Delta_1^2+\Delta_2^2)=0,$$

with ν a product of distinct rational primes, Δ_1 and Δ_2 relatively prime, $\Delta_1^2+\Delta_2^2=\sigma^2\rho$ with $\rho>1$ a product of distinct rational prime integers.

3. Mr. D. H. Lehmer: *On numbers associated with the divisors of an integer.*

Associated with every positive integer n is its set of distinct divisors, which we designate by δ_1. There is also a set of divisors associated with each number of the set δ_1. In this way the set δ_1 gives rise to a new set δ_2 of numbers (not all distinct) which are in fact the divisors of the divisors of n itself. The set δ_2 generates in the same way a set δ_3 and so on. The elements of the set δ_r we call the rth divisors of n. If we take proper divisors instead of divisors we obtain similarly a set d_r which we call the set of rth proper divisors of n. The study of the connection between the same arbitrary function of δ_r and d_r leads to certain inversion formulas which are applied to the determination of formulas for the product and sum of the kth powers of the rth divisors of n. The above definition of δ_r is meaningless if r is negative, but if negative values of r are used in the formulas for the above sums we obtain a sequence of functions of n among which are Euler's totient function $\phi(n)$ and Merten's inversion function $\mu(n)$, or more generally Jordan's function $J_k(n)$. The present investigation can be considered either as a generalization of Jordan's function, or as a study of the sequence of functions $f_r(n)$ connected by the relation $f_{r+s}(n)=\sum f_s(\delta_r)$, where $f_0(n)=n^k$.

4. Mr. D. H. Lehmer: *A generalization of Lucas' functions U_n and V_n.*

The functions $U_n=(a^n-b^n)/(a-b)$ and $V_n=a^n+b^n$, where a and b are the roots of $x^2\times Px+Q=0$, $(p, Q)=1$, have been studied by Lucas, Carmichael, and others. It is shown in this paper that a more homogeneous theory results if P is replaced by R, $(R, Q)=1$, and if U_n and V_n are considered as functions of R and Q. Although U_n and V_n may contain the factor R in this theory, nearly all of Lucas' theorems apply with a slight modification. Some parts of the present theory, however, have no counterpart in Lucas' theory. Laws governing the appearance and repetition of a prime in the series U are similar to those of Lucas. In developing these laws one is led to a new generalization of Fermat's theorem and Euler's totient function. New generalizations of Wilson's theorem result from a discussion of certain special cases of the series U. Associated with U_n expressed as a function of R and Q is a certain irreducible binary form G_n which represents numbers whose factors

are of the forms $kn \pm 1$. The connection between G_n and Sylvester's cyclotomic functions is established. A transformation is discussed which replaces a and b by their rth powers. The inverse of this transformation leads to a new extension of the theory of quadratic and higher residues. This new theory is applied to a discovery of necessary and sufficient conditions for the primality of large numbers of certain types.

5. Dr. C. C. Craig (National Research Fellow): *The semi-invariants and moments of incomplete normal and Type III frequency functions.*

The answers to the following questions depend on a knowledge of the moments or of the semi-invariants of incomplete frequency functions. (1) How can a truncated frequency distribution be completed? (2) The method commonly used for finding the moments or semi-invariants of an index, y/x, is really valid only if the range of x is limited; how do results obtained in this way differ from those in which the range for x is unrestricted? (3) How would the characteristics of the best possible sample from an infinite population differ from those of the parent? To the previous results in the case of the incomplete normal frequency function, the present paper adds direct expressions for the semi-invariants, a simple recursion formula for the moments, and expressions for the moments in polynomials with the range as argument. The Type III curve of Pearson is similarly studied. The direct expressions for the semi-invariants are expressed by a kind of generalized Laguerre polynomials involving the range and skewness in the argument. Simple recursion relations for the moments are found. Numerical examples and selected tables are appended.

6. Mr. W. L. Langdon and Professor Oystein Ore: *Sheppard's correction and semi-invariants.*

The use of semi-invariants for the characterisation of statistical distributions has for many applications a distinct advantage. This paper deals with the problem of correcting semi-invariants, and a very simple formula for the corrected semi-invariants has been established.

7. Mr. W. R. Thompson: *The powers of a given rational prime which may divide exactly the discriminant of an algebraic field of nth degree.*

For a given rational prime, p, Ore (Mathematische Annalen, vol. 96 (1926), pp. 313–352) has shown, in a given algebraic field of nth degree whose discriminant is d, the dependence upon the prime ideal decomposition of p of the exponent, E, such that d is exactly divisible by p^E. He has given the maximum value of E for such a field which he designates by $N(n, p)$. He suggests the interest of discovering what other values E may have. The author presents the following complete solution of this problem. Let $E_p^{(n)}$ be the set of possible values of E for fields of nth degree, and α be a positive rational integer. Then for $p>2$ and $n=p^\alpha$, $E_p^{(n)}=0, \cdots, N(n, p)$ except

$\alpha p^{\alpha-1}-1$ for $\alpha>1$, $p>2$, and $n=p^{\alpha}+1$, $E_p^{(n)}=E_p^{(n-2)}$, for $p>2$ in *every other case*, $E_p^{(n)}=0\cdots$, (N, p), and if $p=2$ the result is the same as the above except that the number 1 is never a component of $E_2^{(n)}$.

8. Dr. A. K. Mitchell: *The derivation of tensors from tensor unctions.*

In this paper, the author first outlines the method by which E. Schroedinger (Annalen der Physik, vol. 82 (1927), p. 265) obtains a rule for deriving a tensor from an invariant tensor function. It is pointed out that Schroedinger's proof applies only for euclidean space. A proof, holding for any metric, is then given of the theorem that if ϕ is any invariant function of a tensor its derivative with respect to a component of this tensor is itself the component of a tensor, and also that the derivative of a tensor function of a tensor produces a tensor of higher rank. In the second part of the paper this theorem is applied to the invariants of a mixed tensor of rank 2. The invariants of these derived tensors are found as functions of the original invariants. Finally the theorem is used to give immediate proofs of some of the elementary facts in algebraic invariant theory, as that the discriminant of an n-ary quadratic form is an invariant of weight 2.

9. Mr. Samuel Borofsky: *Expansion into infinite products of functions defined by Dirichlet series.*

In this paper, it is proved that a function defined in a half-plane by a Dirichlet series

$$1+b_1e^{-\mu_1 z}+\cdots+b_ne^{-\mu_n z}+\cdots,$$

where lim sup $(\log n/\mu_n)$ is finite, can be expressed in some half-plane as an infinite product $\pi(1+a_ne^{-\lambda_n z})$, where $0<\lambda_1<\cdots<\lambda_n<\cdots$, $\lim \lambda_n=\infty$, and lim sup $(\log n//\lambda_n)$ is finite, the λ's being linear combinations of the μ's with positive integral coefficients. This generalizes the corresponding theorem of Ritt for power series.

10. Mr. F. H. Miller: *An application of the method of parameters to linear partial differential equations.*

In this paper it is shown that the method of variation of parameters, widely employed in determining the complete primitive of a linear ordinary differential equation whose complementary function is known, may also be used in connection with certain linear partial differential equations of the second order. Conditions under which the method is applicable are considered, and the connection between it and the usual Laplace transformation is discussed.

11. Professor J. F. Ritt: *On a certain ring of functions of two variables.*

We deal with functions (1): $a_1(x)\alpha_1(y)+\cdots+a_n(x)\alpha_n(y)$, with the a's and α's analytic, but otherwise arbitrary. The product of two functions of this type is a function of similar type. One may thus propose to establish

a factorization theory for functions (1). It is understood, in what follows, that the functions (1) are so expressed that the a's in them, and also the α's, are linearly independent. It is proved that if $A=BC$, where A, B, C are of type (1), every a in B or C is an algebraic combination of the a's in A, if one makes exception of an arbitrary multiplicative function; similarly for the α's. It follows that a function $a(x)\alpha(y)$ cannot be the product of two functions (1) each of which has more than one term. Also, the general factorization problem becomes a problem in algebraic functions of several variables. It is shown that $1+xy+\phi(x)\psi(y)$, with ϕ and ψ transcendental, is not the product of two functions (1), each of more than one term.

12. Professors Philip Franklin and C. L. E. Moore: *Systems of linear partial differential equations.*

In this paper the authors discuss the solution of a pair of linear partial differential equations of the second order, and obtain analytically a result of Segre's, that the geometric interpretation of the solution is either a locus in 3-space, or the developable surface of a twisted curve. Some extensions to other systems are made.

13. Professor J. D. Tamarkin: *A lemma of the theory of differential systems.*

In this paper, the author states and proves a new lemma on linear differential systems with Lebesgue integrable coefficients, from which results that have been given by certain other authors can be derived in a simple manner. In particular, the results of Miss Whelan (this Bulletin, vol. 35(1929), pp. 105–125), and the results of a paper by W. M. Whyburn (see this Bulletin, vol. 36 (1930), p. 59), can be derived from the lemma. This paper and that of W. M. Whyburn will appear in the February issue of this Bulletin.

14. Dr. P. M. Swingle: *Generalized indecomposable continua.*

In this paper two of the more useful definitions of indecomposable continua are generalized. A number of theorems are proved concerning the sets thus defined, and the impossibility of generalizing another of these definitions in an obvious manner is shown.

15. Dr. Jesse Douglas: *Various forms of the fundamental functional in the problem of Plateau and its relation to the area functional.*

The writer's work on the problem of Plateau (see abstracts in this Bulletin, vol. 33 (1927), pp. 143, 259; vol. 34 (1928), p. 405; vol. 35 (1929), p. 292) is based on considering all possible representations $x_i=z_i(\theta)$; $i=1, 2\cdots, n$; of the given contour as topological image of the unit circle, and reducing the problem to finding the representation g which minimizes the functional

$$A(g) = -\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \log|\sin\tfrac{1}{2}(\theta-\phi)|\sum_i dg_i(\phi).$$

In the present paper the following new forms are found for $A(g)$:

$$(1) \qquad A(g) = \frac{1}{16\pi}\int_0^{2\pi}\int_0^{2\pi} \sum_i [g_i(\theta)-g_i(\phi)]^2[\sin\tfrac{1}{2}(\theta-\phi)]^{-2}d\theta d\phi,$$

$$(2) \qquad A(g) = \frac{\pi}{2}\sum_{m=1}^{\infty} m\left(\sum_i a_{im}^2 + \sum_i b_{im}^2\right),$$

where a_{im} and b_{im} denote the Fourier coefficients of $g_i(\theta)$. Sufficient conditions are established for the convergence of the infinite processes involved. These formulas prove $A(g)$ to be always positive. The form (2) subsumes the problem under the theory, initiated by Hilbert, of quadratic forms in a denumerable infinity of variables. The functional $A(g)$ is *different* from the area of the harmonic surface determined by the boundary values $g_i(\theta)$. However, these two functionals take their minimum values for the same representation g, and these minimum values are equal. Thus the functional $A(g)$ serves the same purpose as the area functional while it is decidedly simpler of expression and treatment. The specific advantage of the author's method in the problem of Plateau lies precisely in the substitution of $A(g)$ for the more complicated area functional.

16. Dr. Jesse Douglas: *A general formulation of the problem of Plateau.*

In this paper methods are developed for treating the problem of Plateau in the following general formulation. *Given k contours in n-dimensional euclidean space. To determine a minimal surface bounded by the k contours (and with no other boundaries) and having a prescribed topological structure, that is, genus and two-sidedness or one-sidedness. This minimal surface is, further, to be represented as a conformal image of a canonical configuration of the same topological structure.* The canonical configuration may always be represented as a region bounded by circles; we consider all possible ways of representing the contours as topological images of these circles:

$$x_i = g_i^{(1)}(\theta),\ x_i = g_i^{(2)}(\theta),\ \cdots,\ x_i = g_i^{(k)}(\theta).$$

The problem is then reduced to the minimization of a certain functional $A(g^{(1)}, g^{(2)}, \cdots, g^{(k)})$, generalizing $A(g)$ of the previous paper. The expression for this functional involves, in the place of the trigonometric functions occurring in $A(g)$, those functions which belong to a Riemann surface of the topological structure prescribed for the required minimal surface. For instance, for a minimal surface bounded by two given contours and to be represented conformally on a circular ring, the expression for $A(g^{(1)}, g^{(2)})$ involves elliptic functions, having periods 2π, $2p(-1)^{1/2}$, where p denotes the logarithm of the ratio of the radii of the circular ring. The existence of the required minimal surface depends on the existence of a real root for a certain definite equation in p. These ideas extend to the general case, where the moduli of Riemann intervene in place of the parameter p.

17. Professor O. E. Glenn: *Integral invariants and secular perturbations.*

It is proved in this paper that any planetary orbit perturbed from the true elliptical form by attractions of other planets moves upon a trajectory which may be regarded as the free orbit of a properly determined central force. The transformations which give the perturbed orbit from the corresponding ellipse are made the basis of a theory of perturbations considered from the standpoint of integral invariants.

18. Dr. T. H. Gronwall: *A formula in geometrical optics.*

It is shown that the double surface integral giving the total radiation from a surface S upon a surface S_1 may be transformed into a double line integral over the boundaries C and C_1. In the special case where S and S_1 lie in two parallel planes, the limit of the integral when the planes coincide is πA, where A is the area which S and S_1 have in common when superposed.

19. Professor James Pierpont: *On Cayley's definition of non-euclidean geometry.*

In his sixth memoir on quantics (1859), Cayley incidentally laid the foundations of a projective definition of distance and angle. Whether he saw the relation of his work to that of Lobachevsky and Bolyai is not known. At any rate, the foundations for a simple and complete theory of non-euclidean geometry (in the older sense) are there. The author has developed the implications of this paper, and wishes to show that this treatment of the subject is superior in some if not in all ways to the traditional method of Klein.

20. Professor Edward Kasner: *The ∞^3 plane sections of an arbitrary surface.*

The plane sections of an arbitrary surface form a triply infinite system of importance in projective differential geometry. By projection (orthogonal or central) on a fixed plane a related system is obtained whose characteristic geometric properties are studied. The first property, that the focal locus is always a circle, is the same as for dynamical trajectories (see the author's Princeton Colloquium Lectures, p. 10, or Transactions of this Society, vol. 7 (1906), p. 405), but the others are quite different. The ∞^4 spherical sections of a surface, fundamental in inversion geometry, are discussed briefly.

21. Professor W. A. Wilson: *On the Phragmén-Brouwer theorem.*

The purpose of this note is to give an elementary demonstration of the Phragmén-Brouwer theorem, for spaces other than the euclidean plane, which does not require a knowledge of the theory of dimensionality as a basis. The theorem is shown to be true in any metric space satisfying these requirements: (a) every bounded set has at least one limiting point; (b) if a and b are any two

points, d is the distance between them, and k is any positive number less than d, then there is one and only one point c whose distances from a and b, respectively, are k and $d-k$. The proof is based upon the validity of the theorem in the euclidean plane.

22. Professor W. A. Wilson: *A property of certain hyperspaces.*

Let M be a metric space and H the hyperspace whose points or elements are the bounded closed sets of M, with the distance between any two elements defined as in Hausdorff's *Mengenlehre* (pp. 145–150). This paper shows that, if every bounded set in M has at least one limiting point and M is everywhere locally connected, then H enjoys these properties. Consequently we have, as a generalization of the well known space filling curves, the result that the bounded closed sets of n-dimensional euclidean space (or an n-dimensional closed sphere) may be put in a continuous correspondence with the points of a euclidean ray (or segment).

23. Mr. J. M. Feld and Mr. Philip Newman: *On the representation of analytic functions of several variables as infinite products.*

The object of this paper is to prove (1) that an analytic function

$$f(x,y)=1+\sum b_{mn}x^m y^n$$

can be uniquely represented as an absolutely convergent infinite product $\Pi(1+a_{mn}x^m y^n)$ with constant a's, and (2) that $f(x, y)$ can be represented in the form $\Pi_1^{\infty}(1+P_n)$, where P_n is a homogeneous polynomial in x and y of degree n.

24. Dr. L. W. Cohen (National Research Fellow): *A remark on a system of equations with infinitely many unknowns.*

The non-singular system of equations $x_i+\sum_k a_{ik}x_k=y_i$, where

$$\sum_i\left[\sum_k |a_{ik}|^{p/(p-1)}\right]^{p-1} \quad \text{and} \quad \sum_i |y_i|^p$$

converge ($p>1$), has a unique solution $\{x_k\}$ such that $\sum_k |x_k|^p$ converges. It is shown that the inverse of this transformation is of the same type, that is,

$$\sum_i\left[\sum_k |({}^i_k)|^{p/(p-1)}\right]^{p-1}$$

converges where $({}^i_k)$ is the minor of a_{ik}.

25. Dr. A. B. Brown (National Research Fellow): *Relations between the critical points and curves of a real analytic function of two independent variables.*

In this paper the case where both isolated critical points and curves of critical points appear is treated for the first time by precise methods. Two inequalities and one equation are obtained, involving the isolated points and curves of minimum and maximum, and the "critical points of intermediate type."

26. Professor C. J. Coe: *Exterior motion in the restricted problem of three bodies.*

This paper treats of that case of the restricted problem of three bodies in which the particle moves outside of the closed outer oval of zero relative velocity. Necessary and sufficient conditions are first established that the outer oval be closed, both for a given ratio of the two finite masses and independently of this ratio. The exterior orbit of the particle is first discussed for the neighborhood of its points of contact with the closed outer oval, the nature of these cusps in the rotating plane being completely determined. Next are developed two different sets of sufficient conditions that the particle recede to infinity, and an extension is made of Koopman's discussion of the areal velocity in fixed space for distant portions of the plane. There follow four theorems treating of the angular velocity of the particle, the most striking result being that for orbits not extending to infinity, the motion in the rotating plane can be direct only within a narrow ring surrounding the closed outer oval. Following this are proved five theorems on the total angular displacement of the particle showing, for instance, that it can never advance as much as 60° in the rotating plane.

27. Professor G. A. Parkinson: *Pairs of curves in an S_n.*

This paper discusses pairs of curves in a flat n-dimensional space. In the first part of the paper it is shown that all of the more important properties of Bertrand curves carry over from euclidean 3-space to this more general space, either directly or in slightly modified form. In the subsequent parts of the paper are discussed pairs of curves which have a common pth ($p>2$) normal; pairs of curves for which the principal normal of one lies in the osculating plane of the other; pairs of curves for which the principal normal of one lies in the plane of the principal normal and binormal of the other; and involutes of curves.

28. Professor H. M. Gehman: *Centers of symmetry in analysis situs.*

In this paper is given first a definition of a center of symmetry (in the sense of analysis situs) of a point set lying in an n-dimensional euclidean space. A property which holds true in this case is then used to define a center of symmetry of a point set when the set itself is considered as a space. The principal result of the paper is a characterization of a simple continuous arc from A to B as a closed (or a regular) point set M containing A and B and at least one more point, such that (1) every point of $M-A-B$ is a center of symmetry of M, and (2) A is a non-cut point of M. This definition of an arc, like Sierpinski's, does not explicitly assume that the set M is connected.

29. Mr. N. C. Fisk: *An investigation of surfaces in euclidean 4-space by means of 3-vectors.* Preliminary report.

The properties of a surface in 3-space may be investigated by studying the change in the tangent plane corresponding to an arbitrary displacement

on the surface. In this case the tangent plane is uniquely represented by the unit normal vector. The tangent plane of a surface in 4-space may be represented uniquely by a pair of unit 3-vectors, each in a separate 3-space. This paper studies the properties of a surface in 4-space by investigating the changes in each of a pair of unit 3-vectors corresponding to an arbitrary displacement on the surface. The changes in the unit 3-vectors give rise to a pair of quadratic differential forms, the generalization of the second fundamental form for a surface in 3-space. From the quadratic forms four scalar invariants are obtained.

30. Professor H. T. Davis and Mr. V. V. Latshaw: *Polynominal approximation by the method of least squares.*

This paper, following a suggestion of Karl Pearson, is devoted to the simplification of the method of least squares as applied to the problem of fitting polynomials to data with arguments arranged in an arithmetic progression. The problem resolves itself into the calculation of symmetric determinants and their first minors, the elements of which are the polynomials in p, $S_n(p)$, representing the Bernoulli sums,

$$S_n(p) = 1^n + 2^n + \cdots + p^n.$$

The mathematical interest of the paper centers about two empirical discoveries, (a) that the symmetric determinants, the elements of which are the Bernoulli sums, are completely factorable into rational factors; (b) that each first minor contains as a factor the first minor of lowest degree in p. Although an assumption is made that the data contain an odd number of items, generality is restored to the results by means of a transformation on the moments.

31. Mr. J. R. Abernethy: *A general explicit formula of numerical integration.*

This formula is the most general obtainable for the expression of the approximate value of the definite integral of a function, $f(x)$, as a linear combination of values of the function and its derivatives, containing as special cases all formulas which give exact results if applied to polynomials of degree $n-1$, where n is the number of values appearing in the formula. The choice of values is only limited by the restriction that there be at least $v+1$ values of $f^s(x)$ $(s=0, 1, \cdots, v; v=1, 2, \cdots, n-1)$. The remainder term is a linear combination of divided differences with repeated arguments.

32. Professor H. A. Simmons: *A calculus of variations problem whose extremals are parabolas.*

Suppose light is traveling in a plane medium whose index of refraction n at any point is directly proportional to the square root of the distance of the point from a fixed line of the plane, and let it be required to find the path in this medium along which light will pass from one point to another in the shortest time. This problem requires a minimum for the integral

$$I = \int_{x_1}^{x_2} y^{1/2} (1+y')^{1/2}\, dx.$$

(1) The extremals for this problem are the parabolas, the one-parameter family of which through any point $1,(x_1,y_1)$, has the equation

$$y=y_1+\left[(x-x_1)^2+4c(x-x_1)\ (y_1-c^2)^{1/2}\right]/4c^2.$$

(2) The envelope, G, of this family is a parabola. (3) Through any point $2,(x_2,y_2)$, above G and not in the same vertical line with 1 there pass exactly two parabolas of the family through 1. (4) One of the two parabolas just mentioned provides a relative minimum for I, while the other touches G at a point for which $x_1<x<x_2$ and therefore does not satisfy Jacobi's condition. (5) There is a discontinuous solution consisting of segments of the straight lines $x=x_1$, $x=x_2$, $y=0$. (6) There is a curve H, defined by an elliptic integral, which plays the role here that MacNeish's curve plays in the catenary problem.

33. Professor V. B. Teach: *The Hamilton-Jacobi theory for the problem of Lagrange in parametric form.*

The problem of Lagrange in parametric form is that of finding in a class of arcs $y_i=y_i(t)$, $(i=1,\cdots,n)$, passing through two fixed points and satisfying a system of differential equations

$$\phi_\beta(y,y')=0, \qquad (\beta=1,\cdots,m<n-1),$$

an arc which minimizes the definite integral $\int_{t_0}^{t_1}f(y,y')dt$, the symbols y and y' representing the sets $(y_1,\cdots,y_n)$ and $(y_1,\cdots,y_n)$ respectively. In this paper the theory of the parametric problem in (y)-space is deduced from the theory of the corresponding non-parametric problem in (t,y)-space, a differential equation $\phi(y,y')-1=0$ defining the parameter t being adjoined to the system $\phi_\beta=0$ for the latter problem. First necessary conditions are obtained and shown to be independent of the choice of parameter. The structure of those special Mayer fields in (t,y)-space of interest to the parametric problem is next discussed. A canonical form for the differential equations defining extremals and the Hamilton-Jacobi partial differential equation are determined and their properties studied with reference to the theory of fields.

34. Dr. W. D. Baten: *Simultaneous treatment of discrete and continuous probabilities by use of Stieltjes integrals.*

This paper presents several theorems pertaining to the probability that certain functions lie within certain intervals. The first theorem is a generalization of Markoff's Lemma, which is proved for the discrete and continuous cases by use of the accumulative frequency function and Stieltjes integrals. Tchebycheff's theorem is obtained as a corollary to a very general theorem, the proof of which is based on the first theorem. Other corollaries are given. Three theorems due to Guldberg, concerned with the probability that a non-negative chance variable be less than certain functions of the expected value of the variable, are proved for the discrete and continuous cases by employing accumulative frequency functions and Stieltjes integrals. This is the first time, as far as the writer knows, that the discrete and continuous cases for these theorems have been included in a single proof.

35. Professor C. N. Moore: *Gibbs' phenomenon in summable series of Bessel's functions.*

In a previous paper (abstract in this Bulletin, July-August, 1928) the author has shown that the nature of Gibb's phenomenon in the case of convergent series of Bessel's functions can be readily inferred from an asymptotic formula for the coefficients of these series obtained by him in 1911. In the present paper he shows how a more general asymptotic formula (see Transactions of this Society, vol. 21 (1920), p. 110) serves to answer the the question as to the appearance of Gibbs' phenomenon when the same series are summed by Cesàro means.

36. Mr. H. C. Chang: *Some transformations of linear partial differential equations.*

In the first part of the paper a generalization of Cauchy-Riemann equations given recently by K. P. Williams is carried further. A system of two linear homogeneous partial differential equations with constant coefficients involving two independent variables and two functions is considered. It is asked when such a system can be transformed into one equivalent to the Cauchy-Riemann system by subjecting the variables as well as the functions to linear transformations with constant coefficients. It is found that a necessary and sufficient condition is the inequality

$$4(B'C'-A'D')\,(BC-AD)>(AD'+A'D-BC'-B'C)^2.$$

The second part is devoted to a generalization of the problem to the case of four equations involving three variables and three functions. The system into which it is required to transform such a system consists of three equations expressing the fact that the rotation of a vector vanishes and of one equation saying that its divergence vanishes. A necessary and sufficient condition is given here by seven equalities and two inequalities.

37. Dr. W. T. Reid: *Note on an infinite system of linear differential equations.*

In this paper is treated the infinite system of linear differential equations of the first order which we may write in vector form (1): $y'=A(x)y$, where $A(x)\equiv(A_{ij}(x))$ is an infinite square matrix each of whose elements is Lebesgue summable on X: $0\leqq x\leqq 1$ and such that for all real quantities $\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_n$, and every integer n,

$$\left|\sum_{i=1}^{n}\sum_{i=1}^{n}A_{ij}(x)\xi_i\eta_j\right|\leqq\phi(x)\,\Big(\sum_{i=1}^{n}|\xi_i|^{[p/(p-1)]}\Big)^{[(p-1)/p]}\Big(\sum_{i=1}^{n}|\eta_j|^p\Big)^{1/p},$$

where p is a real quantity greater than unity and $\phi(x)$ is a summable function on X. By definition a solution of (1) is a vector $y(x)\equiv(y_i(x))$ which satisfies the equation "almost everywhere" on x and such that

$$\Big(\sum_{j=1}^{\infty}|y_i(x)|^p\Big)^{1/p}$$

is bounded uniformly on X by a finite constant. The solutions of (1) are shown to have properties corresponding to those already proved by the author for the case $p=2$ [see this Bulletin, vol. 35 (1929), p. 147 and p. 454].

38. Mr. A. E. Ross: *On representation of integers by definite*

ternary quadratic forms whose determinant contains no square factors.

In this paper we study the problem of representation of integers by indefinite ternary quadratic forms. Meyer (Journal für Mathematik, vol. 108 (1891), pp. 125–139) gave a criterion for equivalence of such forms and employed the latter to obtain sufficient (but not necessary) conditions that they represent a given integer m. A. W. Markoff (Mémoires, Académie des Sciences, St. Petersbourg, (8), vol. 23 (1909)) constructed a table of indefinite ternary quadratic forms of positive determinants $D \leqq 50$. L. E. Dickson, in an unpublished manuscript, noted regularity in form of the entries of the table and employed a Dirichlet method to obtain complete theorems on representation of integers by forms of determinant $-D$, where $D \leqq 83$ and is either a prime, double a prime, the product of two distinct primes, or double such a product. By Meyer's criterion for equivalence and ideas suggested by Professor Dickson's method this paper solves the problem of representation of integers by indefinite ternary quadratic forms whose determinant is any integer containing no square factors. Theorems proved by Professor Dickson are found to hold for any D of the same type.

39. Professor M. H. Ingraham: *Nilpotent and indempotent algebras of infinite order.*

Many theorems concerning the nilpotent subalgebras and the idempotent elements of a linear associative algebra of finite order are not available for the infinite case. A definition of transfinite powers of an algebra is given and much of the theory of nilpotent subalgebras reestablished for the case in which the algebra has an index, finite or transfinite. A sufficient condition for the existence of such an index is that the cardinal number of the algebra be an "aleph number." An algebra may not be nilpotent and yet contain no idempotent element. Much of the theory based on idempotent elements may, however, be replaced by one based on idempotent subalgebras. These idempotent subalgebras are of several types corresponding roughly to types of idempotent elements in the finite case.

40. Professor V. C. Poor: *Residues of polygenic functions.*

This paper contains a definition for the residue of a polygenic function; this definition may be thought of as a generalization of the definition for the residue of an analytic function at a point; the contour chosen for the path of integration is a circle of radius r; the residue is then defined as the limit of the integral of the function around the circle as r approaches zero. Among the several simple cases treated is the residue of the reciprocal of $az+b\bar{z}$. Here the poles may be distributed along a straight line through the origin. An application of the definition is also made to the derivation of the "areolar" derivative of Pompieu.

41. Dr. L. M. Biumenthal: *Permutable functions in the complex domain.*

Pérès has shown that the function $G(x, y)$ permutable with a function

$F(x, y)$ of the first order and in canonical form can be represented by a transformation $\Omega(\lambda)$ that preserves composition i.e., if $G=\Omega(\lambda)$, $H=\Omega(\mu)$, then $GH=\Omega(\lambda)\Omega(\mu)=\Omega(\lambda\mu)$, where by GH is meant the Volterra product of G and H (Volterra et Pérès, *Leçons sur la Composition*, p. 58). Those transformations may be written in the form $(1^0+m)\lambda(1^0+n)$, where $m(x, y)$ and $n(x, y)$ are any two functions such that $(1^0+m)(1^0+n)=1^0$. For functions $F(x, y)$ of order $n+1$ (n, a positive integer >1) it is found to be of advantage to leave the field of real variables and to consider a function $F(z, w)$ analytic in the complex variables z, w. This paper seeks to avoid the restrictions of analyticity and investigates the existence of functions of the first order permutable with a non-analytic function F of order greater than two.

42. Professor I. A. Barnett: *Sphere geometry and conformal transformations in function space.*

In order to obtain a simple analytic representation of conformal transformations in function space, the author finds it desirable to define sphere coordinates in a manner analogous to that used in n-space. He is then able to express the general conformal group in ordinary function space as a linear transformation in sphere function space. As an aid in the discussion of the group properties, the author expresses every conformal transformation as a Fredholm transformation. Infinitesimal conformal transformations are next introduced and it is shown that every infinitesimal conformal transformation generates a one-parameter group of conformal transformations.

43. Professor H. A. Simons: *The first variation of a triple integral in the case of variable limits.*

This paper contains an extension of a portion of an article by the present author on the first and second variations of a double integral in the case of variable limits (See Transactions of this Society, April, 1926). Here, instead of considering the double integral $\iint_{A_0} f(x,y,z,p,q)dxdy$, which was studied in the article just mentioned, we consider the triple integral $\iiint_{V_0} f(x,y,z,t,l,m,n)\,dxdydz$. We extend our previous methods and obtain analogs of the Euler condition and the transversality condition which were found in the cited article. The hypersurface $t=t(x,y,z)$ of three dimensions, in euclidean S_4, is given a variation of the type $t=t(x,y,z)+a\tau(x,y,z)$ and these varied hypersurfaces are required to have real closed intersections with a fixed hypersurface $\phi(x,y,z,t)=0$.

44. Professor M. W. Haskell: *Autopolar configurations in the plane and in space.*

The paper is mainly an account of the author's investigations of methods for deriving curves and surfaces that are autopolar in a finite number of ways. One method is an amplification of Appell's paper in the Nouvelles Annales, (3), vol. 13 (1894). A second method, in three dimensions, employs covariants of transformations of line-coordinates.

45. Professor W. M. Whyburn: *Generalized Riccati differential equations.*

The paper considers the set of differential systems defined by the recurrence relations

$F_0(Z)=Z(x)$, $F_i(Z)=d\{F_{i-1}(z)\}/dx+F_{i-1}(Z)\cdot Z=f_i(x)$, $(i=1, 2, \cdots,)$,

where $Z(x)$, $f_i(x)$, and $F_i(Z)$ are square matrices of n^2 elements and x is a real or complex variable. This set of differential equations is shown to be closely related to systems of linear first order differential equations. Important special cases are solved and these solutions shown to represent such important classes of functions as the rational functions, the exponential functions, etc.

46. Professor W. M. Whyburn: *On related difference and differential systems.*

The passage to the limit from difference to differential systems as given by the author (American Journal of Mathematics, vol. 51, pp. 265–286) is shown to be uniform on the interval of definition and to be valid for every method of defining the coefficients of the difference systems so long as these coefficients are uniformly bounded by a summable function and approach the corresponding coefficients of the differential systems almost everywhere. Estimates of the degree of approximation of the solutions of the differential systems by the solutions of the difference systems are immediate consequences of the established uniformity.

47. Professor E. R. Hedrick: *Generalizations of Liouville's theorem and allied theorems.*

In this paper, the author shows first that the usual theorem that the absolute value of a function of a complex variable cannot have a maximum is readily extensible to functions that are non-analytic, provided the jacobian of the corresponding real transformation is different from zero, that is, provided there is no branch point within the region considered. It is then shown that Liouville's theorem can be extended, under the same hypotheses as those usually stated for analytic functions, if the function approaches a definite limit as the variable becomes infinite. The fundamental theorem of algebra, and generalizations of it, follow. Finally, the theorems can all be extended readily to n dimensions; the statements for three dimensions are given.

48. Dr. A. R. Williams: *On certain quartic surfaces which are the envelopes of two systems of conicoids.*

This paper deals with the envelope of a one-parameter family of ellipsoids or hyperboloids of revolution that have one focus fixed while the other moves on a curve C. The characteristics of which the envelope is composed are conics. If the length of the major axis of the conicoid does not vary, the planes of the characteristics are perpendicular to C. If C is a circle and the major axis is constant, the envelope is a quartic surface which is also enveloped by two other systems of conicoids of revolution with variable major axes. Such surfaces have useful optical properties.

49. Professor E. T. Bell: *Three degeneracies in the theory of ternary quadratic forms.*

All values of n are found for which the number of representations of n in one or other of the three forms $xy+yz+zx$, $xy+2yz+2zx$, $xy+yz+2zx$, is a polynomial in the real divisors of n alone. This paper will appear in the Annals of Mathematics.

50. Professor E. T. Bell: *A triple of ternary quadratic forms.*

Beyond *pure* quadratic ternaries, for numbers of representations $ax^2+by^2+cz^2$, only one ternary quadratic, namely, $xy+yz+zx$, has been discussed, notably by Liouville, Bell, Mordell, and Gage. In this paper is presented a complete discussion concerning numbers of representations in each of the three closely connected forms (1, 1, 1), (1, 1, 2), (1, 2, 2), where $(a, b, c) \equiv axy+byz+czx$.

51. Professor E. T. Bell: *On certain quinary and senary quadratic forms.*

Certain general principles, too detailed for concise abstract, are laid down for enumerations of representations in n-ary quadratic forms and are applied to the cases $n=5, 6$. In general, an odd and its consecutive even number of variables are treated together in the theory, which is new and which has but slight connection with classical theories.

52. Professor D. N. Lehmer: *On Franklin magic squares.*

The author has found a generalization of Franklin's method of constructing magic squares of even order which applies to oddly even as well as to evenly even squares. Enumeration formulas have been developed. The number of 4×4 squares has been determined to be at least 6144 instead of 4352 as determined by Andrews. They all come from four types of Franklin squares.

53. Mr. W. H. Ingram: *A system of integral equations implying and implied by a system of first order linear differential equations.*

Bounitzky has shown how, given a system of first order linear differential equations satisfying a set of boundary conditions for certain values of a parameter, a system of integral equations of the second kind can be constructed whose solutions satisfy the differential equations and the boundary conditions. It has not been shown that no other system of integral equations could have a similar property. It is now shown that the system, $v'+(r(x)+\lambda l(x))u=0$, $u'+s(x)+\lambda c(x))v=0$, with the boundary conditions $a_1v(0)+b_1u(0)=0$, $a_2v(1)+b_2u(1)=0$, implies and is implied by the system of integral equations

$$v=\lambda\int_0^1\{K_{11}(x,\xi)\,v(\xi)+K_{12}(x,\xi)\,u(\xi)\}\,d\xi,$$

$$u=\lambda\int_0^1\{K_{21}(x,\xi)\,v(\xi)+K_{22}(x,\xi)\,u(\xi)\}\,d\xi,$$

where $K_{11}(x,\xi)=c(\xi)\,G_{11}(x,\xi)$, etc., in which the G's are Green's functions.

54. Mr. W. H. Ingram: *Systems of integral equations and their Fredholm solutions.*

The system of integral equations of the previous paper has, formally, two solutions of Fredholm type for v and two for u. From the fact that the Green's functions can be expressed as the products of fundamental solutions (Grundlösungen of the Schmidt-Hilbert theory) in the form $G_{11}(x, y) = -v^{**}(y)v^{*}(x)$, $G_{12}(x, y) = -u^{**}(y)v^{*}(x)$, for $0 \leqq x < y$, or $G_{11}(x, y) = -v^{*}(y)v^{**}(x)$, $G_{12}(x, y) = -u^{*}(y)v^{**}(x)$, for $y \leqq x \leqq 1$, and from the relations of symmetry $G_{11}(x, y) = G_{11}(y, x)$, $G_{12}(x, y) = -G_{21}(y, x)$, $G_{22}(x, y) = G_{22}(y, x)$, it appears that when constants l_0, l_1 can be so chosen that we have simultaneously $u^{**}(l_0)\, l\,(l_0) = v^{**}(l_1)\, c(l_1)$, $u^{*}(l_0)\, l(l_0) = v^{*}(l_1)\, c(l_1)$, then the solutions for u and v respectively are identical.

55. Mr. W. H. Ingram: *Application of the theory of integral equations to the theory of the electrical transmission line.*

The differential equations are those of the first paper. The Green's functions represent current and potential due to two types of source physically representable by voltaic cells connected to the line by insertion in series at a point or by parallel connection to ground at a point; they are, therefore, experimentally tabulable. The problem of transient oscillations presents certain difficulties due to the fact that the system of differential equations is not self-adjoint. When the line is dissipationless, any given initial current and potential distributions U and V have the expansions $\sum a_i l^{1/2}(u_i + \bar{u}_i)$, $\sum a_i c^{1/2}(v_i + \bar{v}_i)$ respectively, where the a_i's can be determined to minimize the quantity

$$N = \int_0^1 \{ [U - \sum a_i l^{1/2}(u_i + \bar{u}_i)]^2 + [V - \sum a_i c^{1/2}(v_i + \bar{v}_i)]^2 \} dx.$$

In the presence of dissipation (r and s not zero), a similar expansion is obtainable which minimizes (maximizes) the indefinite form

$$M = \int_0^1 \{ [U - \sum a_i l^{1/2}(u_i + \bar{u}_i)]^2 - [V - \sum a_i c^{1/2}(v_i + \bar{v}_i)]^2 \} dx.$$

56. Professor Raymond Garver: *On the transformation which reduces the Brioschi normal quintic to a general principal quintic.*

If the transformation

$$Y = \frac{\lambda + \mu w}{Z^{-1}w^2 - 3}$$

is applied to the Brioschi quintic

$$w^5 - 10Zw^3 + 45Z^2w + k = 0,$$

the transformed equation $F(Y) = 0$ is of the so-called principal form, lacking terms in Y^4 and Y^3. Further, $F(Y) = 0$ can be identified with any principal quintic, with certain minor exceptions, by properly choosing λ, μ, Z. The importance of the transformation depends on this fact, since it follows almost at once that any sufficiently general quintic can be reduced to the Brioschi form. This paper gives a method of forming $F(Y) = 0$ which seems simpler

than the methods which have been given previously, and which allows the whole topic to be presented in a very convenient form.

57. Professor Raymond Garver: *Concerning polynomial functions with certain properties.*

Brioschi has shown the existence, for the general nth degree equation, n odd, of $(n-1)/2$ polynomial functions which possess certain properties making them of value in a number of transformations of equations. His proof, however, is long and yet incomplete. This paper presents what seems to be a more satisfactory proof. The case where n is even is also considered, and it is shown that $n/2$ functions with the desired properties cannot, in general, be set up.

58. Professor Florian Cajori: *Newton's idea of God in the different editions of the Principia.*

Although the first edition of the *Principia* contains no direct expression of the nature of God, Newton later stated that he hoped its contents would lead to a belief in the Deity. The Principia was criticised, on theological grounds, by Bishop Berkeley and by Leibniz. In the second edition Newton added a general scholium giving his idea of God as seen from His works. Interpolations in the third edition stressed the idea that our notions of God are taken more especially from the ways of mankind.

59. Dr. Gordon Pall (National Research Fellow): *On the representations of a constant times a square as the sum of an odd number of squares.*

Stieltjes and A. Hurwitz found expressions as simple divisor functions for the number $r(n^2)$ of the representations of n^2 as the sum of s squares, $s=3$ or 5. This paper gives a general formula for the ratio $r(n^2c)/r(c)$ as a function of divisors, when $s=3$, 5, 7 and when $s=11$, $c\equiv 7 \pmod 8$. When $c=1$, $s=3$, 5, this reduces to Hurwitz's results.

60. Dr. B. W. Jones (National Research Fellow): *On certain irregular ternary forms with cross products.*

In this paper the writer finds that in the neighborhood of 200 reduced positive primitive forms $f=ax^2+by^2+cz^2+2ryz$ are regular, where b, c, and $2r$ have no factor greater than 2 in common and r is odd if b and c are even. This is equivalent to finding what forms $x^2+dy^2+paz^2$ and $x^2+dy^2+2paz^2$ are regular as to multiples of p or $2p$ respectively, where $(-d\,|\,p)=1$, p is a prime, and, in the second case, $d\equiv 3 \pmod 4$.

61. Dr. Clifford Bell: *In- and circumscribed sets of planes to space curves.*

In this paper the number of in- and circumscribed sets of osculating planes to the various 3-space rational quartic curves is determined. It is also shown that there are $4n$ unclosed sets of three osculating planes to the cuspidal quartic, each set of which has its point of intersection on the surface $f(x, y, z, w)=0$

of order n. The points of contact of the first, second, and third osculating planes are represented respectively by A_i, B_i, C_i, the other intersection of the third plane being represented by D_i, and the point on the surface $f(x, y, z, w)=0$ at which the three planes intersect is represented by P_i, $(i=1, 2, \cdots 4n)$. It is proved that the surface $f(x, y, z, w)=0$ can be deformed by a simple magnification of x, y, z, and w in such a way as to contain all the points A_i, and likewise for the points B_i, C_i, and D_i. This is also extended to the analogous curve in n-space.

62. Dr. H. T. Engstrom (National Research Fellow): *On the ideal theory of Zolotarev.*

This paper is a commentary on the theory of Zolotarev based on a translation from the Russian of an exposition by Tchebotarev. The common criticisms of the theory on the grounds of incomplete generality are probably based on the inadequate account in the Fortschritte der Mathematik and are unjustified.

63. Dr. J. D. Elder: *Fourier expansions of doubly periodic functions of the third kind.*

A method, due to Appell, is applied to the determination of the Fourier expansion of the function $F(z)=\theta_0^{n_0}(z)\theta_1^{n_1}(z)\theta_2^{n_2}(z)\theta_3^{n_3}(z)$, where the n's are integers such that $n_0+n_1+n_2+n_3<0$. Sixteen sub-cases are considered, which arise from the possible parities of the n's. For each of these, an explicit formula of expansion is found, valid for general values of the n's of appropriate parities.

64. Dr. Morgan Ward: *The characteristic number of a sequence of integers satisfying a linear recursion relation.*

The author shows that if $(A)_n$ is any sequence of integers satisfying $X_{n+3}\equiv PX_{n+2}-QX_{n+1}+RX_n \pmod m$, and if $m=p_1^{a_1}\cdots p_r^{a_r}$ is the decomposition of m into prime factors, then the characteristic number of $(A)_n$ modulo m is the L. C. M. of its characteristic numbers moduli $p_1^{a_1}, \cdots, p_r^{a_r}$. If $m=p^a$, the characteristic number is of the form $p\mu^b$, where μ is the characteristic number of $(A)_n$ modulo p, and $b\leqq a-1$. The determination of μ is equivalent to the unsolved problem of determining the period of a given mark in a given Galois field. The results obtained may easily be extended to the general case of a recursion relation of order r. (For the definition of the characteristic number, see Carmichael, American Mathematical Monthly, March, 1929.)

65. Professor P. H. Daus: *Note on a table of linear forms.*

In constructing his table of linear forms, used in making his factor stencils, D. N. Lehmer used the fact that if r_1 and r_2 are numbers in a linear form, corresponding to a given D, then r_1r_2 is also a number of the form. This paper is concerned with the question of finding an r, if possible, for a given D, such that all numbers of the form can be obtained from r. It is shown that if $\pm D$ is a prime or twice a prime, such an r can readily be found, using available tables

of primitive roots and the quadratic reciprocity law. By means of the rules established, it is possible to construct a compact table of linear forms.

66. Professor J. V. Uspensky: *A simple proof of the theorem on the limit of probability.*

The fundamental theorem usually called "the theorem on the limit of probability" was proved under very general conditions by Liapounoff. This remarkable proof is based on a very ingenious device consisting in the introduction of an auxiliary contin ıous variable. This auxiliary variable does not appear explicitly in the simplified proof presented in this paper, which makes it comparatively easy and short.

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Meetings of the Society have been fixed at the following times and places:

NEW YORK CITY, February 22, 1930.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than February 1. An invited address will be given by Mr. E. C. Molina on *The theory of probabilities.*

STANFORD UNIVERSITY, CALIFORNIA, April 5, 1930.

Abstracts must be in the hands of Associate Secretary T. M. Putnam, University of California, Berkeley, Calif., not later than March 8.

NEW YORK CITY, April 18-19, 1930.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than March 28. The Committee on Program is arranging a symposium on Differential Geometry, in which Professors Oswald Veblen, L. P. Eisenhart, W. C. Graustein, and others will participate.

CHICAGO, April 18-19, 1930.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than March 28. An invited address will be given by Professor Henry Blumberg on *Methods in point sets and the theory of real functions.*

R. G. D. RICHARDSON, *Secretary of the Society.*

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